Explicit computation of Gross-Stark units over real quadratic base fields

Paul Thomas Young

College of Charleston

December 19, 2012
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This talk represents joint work with Brett Tangedal, UNCG.


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Hilbert’s Twelfth Problem

D. Hilbert, Paris 1900: Given a finite extension field $F$ of $\mathbb{Q}$, give an analytic construction of all abelian extensions $K$ of $F$, using only information from $F$.  

Kronecker, Weber (1877, 1886, 1896, 1908): Every abelian extension $K$ of $F = \mathbb{Q}$ is contained in a cyclotomic field $\mathbb{Q}(e^{2\pi i/n})$ for some positive integer $n$. That is, every abelian extension $K$ of $\mathbb{Q}$ may be obtained from $\mathbb{Q}$ by adjoining values of the analytic function $f(z) = e^{2\pi iz}$ for $z \in \mathbb{Q}$.

Takagi (1920): If $F = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field, then every abelian extension $K$ of $F$ may be obtained by adjoining to $F$ values of $f(z) = e^{2\pi iz}$; the elliptic modular function $j(\tau)$; and/or the Weierstrass $\wp$ function $\wp(z, \tau)$ for $z \in \mathbb{Q}, \tau \in F$.

T, Y (2012): If $K$ is a totally complex abelian extension of a real quadratic field $F = \mathbb{Q}(\sqrt{d})$, then $K$ is generated by $p$-adic exponentials of values of our $p$-adic double log gamma function $G_p, 2(z; (\omega_1, \omega_2))$ for $z, \omega_i \in F$.

Using these $p$-adic functions, we give an effective, efficient algorithm that explicitly constructs $K$ from $F$ analytically ($p$-adic analytically).
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A number field $K$ is a finite (algebraic) extension field of $\mathbb{Q}$, of dimension $n = [K : \mathbb{Q}]$. Associated to it we have:

- The **Galois group** $G = \text{Gal}(K/\mathbb{Q})$ of field automorphisms of $K$.
- If $|G| = [K : \mathbb{Q}]$ then $K$ is a **Galois extension** of $\mathbb{Q}$, and if $G$ is also abelian then $K$ is an **abelian extension** of $\mathbb{Q}$.

- The **ring of algebraic integers** $O_K = \{ \alpha \in K : \exists$ monic $f \in \mathbb{Z}[x], f(\alpha) = 0 \}$.
- The **multiplicative group** $O_K^\times$ of units of $O_K$.

- Every ideal $A$ of $O_K$ has a unique factorization as a product of prime ideals.

- Every nonzero ideal $A$ of $O_K$ has finite index in $O_K$, called the **norm** of $A$, $N_A = |O_K/A|$.

- The **Dedekind zeta function** of $K$, for $\Re(s) > 1$:
  \[
  \zeta_K(s) = \sum_{\text{ideals } A} N_A^{-s} = \prod_{\text{prime ideals } P} (1 - N_P^{-s})^{-1}.
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- Ring $\mathcal{O}_K = \{\alpha \in K : \exists \text{ monic } f \in \mathbb{Z}[x], f(\alpha) = 0\}$ of algebraic integers of $K$. 

Multiplicative group $\mathcal{O}_K^\times$ of units of $\mathcal{O}_K$.

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Dedekind zeta function of $K$, for $\Re(s) > 1$:

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Analytic class number formula

The Dedekind zeta function $\zeta_K$ of $K$ has an analytic continuation to $\mathbb{C}$ except for a simple pole at $s = 1$, with residue

$$\lim_{s \to 1} (s - 1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K \sqrt{|d_K|}},$$

where

$r_1$ is the number of real embeddings of $K$, and $r_2$ is the number of pairs of complex embeddings of $K$; $h_K$ is the class number, which measures how far $\mathcal{O}_K$ is from having unique factorization; $\mathcal{O}_K$ is a UFD $\iff h_K = 1$; the regulator $R_K$ of $K$ is a certain determinant formed from the $r_1 + r_2 - 1$ generators of the torsion-free part of unit group $\mathcal{O}_K^\times$, which measures how "dense" the units are in $\mathcal{O}_K$; $w_K$ is the number of roots of unity in $K$; $d_K$ is the discriminant of $K$. 

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Partial zeta functions

Suppose $K$ is a totally complex abelian extension of a real quadratic field $F$, with $G = \text{Gal}(K/F)$. Choose a prime $(p)$ of $\mathbb{Z}$ which splits as $(p) = p\overline{p}$ in $F$, and such that $p$ splits completely in $K$. Let the set $T$ consist of $p$ together with all infinite primes of $\mathcal{O}_F$ and all finite primes of $\mathcal{O}_F$ which ramify in $K$, and $S = T \cup \{p\}$.

Associated to every $\sigma \in G$ there is a partial zeta function $\zeta_S(s; \sigma)$ defined by

$$\zeta_S(s; \sigma) = \sum_{\sigma \mathcal{A} = \sigma} \mathcal{N}\mathcal{A}^{-s}, \quad (\Re(s) > 1),$$

where the sum is over all ideals $\mathcal{A}$ of $\mathcal{O}_K$ relatively prime to all ideals in the set $S$ and having the specified automorphism $\sigma \mathcal{A} = \sigma$ as its image in $G$ under the Artin map. It has an analytic continuation to all of $\mathbb{C}$ except a simple pole at $s = 1$. 

Cassou-Nogues (1979):

There also exists a $p$-adic partial zeta function $\zeta_S, p(s, \sigma)$ for each $\sigma \in G$ such that $\zeta_S, p(-k; \sigma) = \zeta_S(-k; \sigma)$ for $k \equiv 0 \pmod{p-1}$ and $\zeta_S, p(s, \sigma)$ is $p$-adically analytic on a disc in $\mathbb{C}$ containing $s = 0$. 

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$$
\zeta_S(s; \sigma) = \sum_{\sigma_A = \sigma} N A^{-s}, \quad (\Re(s) > 1),
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\textbf{Cassou-Nogues (1979):} There also exists a $p$-adic partial zeta function $\zeta_{S,p}(s, \sigma)$ for each $\sigma \in G$ such that

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\zeta_{S,p}(-k; \sigma) = \zeta_S(-k; \sigma) \quad \text{for} \quad k \equiv 0 \pmod{p - 1}
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and $\zeta_{S,p}(s, \sigma)$ is $p$-adically analytic on a disc in $\mathbb{C}_p$ containing $s = 0$. 

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Let $F$, $K$, $(p) = p\overline{p}$, $S$, and $T$ be as above, and define the subgroup

$$U_p = \{ \beta \in K^\times : |\beta|_\mathfrak{Q} = 1 \text{ if } \mathfrak{Q} \nmid p \}$$

of $K^\times$. Fix a prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$ lying over $p$ and denote by $x \mapsto x\mathfrak{P}$ the embedding of $K$ into $\mathbb{Q}_p$ corresponding to $\mathfrak{P}$. Then there exists a unique element $\alpha \in U_p$ such that

$$(\sigma(\alpha))^\mathfrak{P} = p^{w_p\zeta_T(0; \sigma)} \exp^{\mathfrak{p}}(\frac{w_p\zeta_S(0; \sigma)}{w_p})$$

for all $\sigma \in G$, and $K(\alpha^{1/w_K})$ is an abelian extension of $F$. In this formula $w_p$ denotes the number of roots of unity in $\mathbb{Q}_p$, and $\exp^{\mathfrak{p}}(x) = \sum k x^k/k!$ is the $p$-adic exponential, convergent for $x \in \mathbb{Z}_p$. It is known that $w_p\zeta_T(0; \sigma)$ is an integer; there are efficient algorithms for computing it (Tangedal, JNT 2007).

This theorem was recently proved (Annals 2011), in the case described above, by Darmon, Dasgupta, and Pollack, without directly constructing $\alpha$. Brett and I give an algorithm using $G_p$, $\mathfrak{p}$ to compute the right side in $\mathbb{Q}_p$, and explicitly gives the irreducible polynomial $f_\alpha \in F[X]$ whose roots are the $\sigma(\alpha)$. 

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$$U_p = \{\beta \in K^\times : |\beta|_\mathfrak{Q} = 1 \quad \text{if} \quad \mathfrak{Q} \nmid p\} \quad \text{of} \quad K^\times.$$ 

Fix a prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$ lying over $p$ and denote by $x \mapsto x_{\mathfrak{P}}$ the embedding of $K$ into $\mathbb{Q}_p$ corresponding to $\mathfrak{P}$. Then there exists a unique element $\alpha \in U_p$ such that

1. $$(\sigma(\alpha))_{\mathfrak{P}} = p^{w_p \zeta_T(0; \sigma)} \exp_p(-w_p \zeta'_{S, p}(0; \sigma)) \quad \text{for all } \sigma \in G,$$ and
Let $F$, $K$, $(p) = p\mathfrak{p}$, $S$, and $T$ be as above, and define the subgroup

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1. $(\sigma(\alpha))_{\mathfrak{P}} = p^{w_p \zeta_T(0;\sigma)} \exp_p(-w_p \zeta_S,p(0;\sigma))$ for all $\sigma \in G$, and
2. $K(\alpha^{1/w_K})$ is an abelian extension of $F$. 

In this formula $w_p$ denotes the number of roots of unity in $\mathbb{Q}_p$, and $\exp_p(x) = \sum k(x^k/k!)$ is the $p$-adic exponential, convergent for $x \in 2p\mathbb{Z}_p$. It is known that $w_p \zeta_T(0;\sigma)$ is an integer; there are efficient algorithms for computing it (Tangedal, JNT 2007). 

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Brett and I give an algorithm using $G$, $2$ to compute the right side in $\mathbb{Q}_p$, and explicitly gives the irreducible polynomial $f_\alpha \in F[X]$ whose roots are the $\sigma(\alpha)$.
Gross’ refined conjecture (1982)

Let $F$, $K$, $(p) = \mathfrak{p}\mathfrak{p}$, $S$, and $T$ be as above, and define the subgroup

$$U_p = \{\beta \in K^\times : |\beta|_\mathfrak{Q} = 1 \text{ if } \mathfrak{Q} \not| p\} \text{ of } K^\times.$$ 

Fix a prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$ lying over $p$ and denote by $x \mapsto x_{\mathfrak{P}}$ the embedding of $K$ into $\mathbb{Q}_p$ corresponding to $\mathfrak{P}$. Then there exists a unique element $\alpha \in U_p$ such that

- $(\sigma(\alpha))_{\mathfrak{P}} = p^{w_p\zeta_T(0;\sigma)} \exp_p(-w_p\zeta'_S,\mathfrak{p}(0;\sigma))$ for all $\sigma \in G$, and
- $K(\alpha^{1/w_K})$ is an abelian extension of $F$.

In this formula $w_p$ denotes the number of roots of unity in $\mathbb{Q}_p$, and $\exp_p(x) = \sum_k (x^k/k!)$ is the $p$-adic exponential, convergent for $x \in 2p\mathbb{Z}_p$. It is known that $w_p\zeta_T(0;\sigma)$ is an integer; there are efficient algorithms for computing it (Tangedal, JNT 2007).
Gross’ refined conjecture (1982)

Let $F$, $K$, $(p) = p\mathfrak{p}$, $S$, and $T$ be as above, and define the subgroup

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This theorem was recently proved (Annals 2011), in the case described above, by Darmon, Dasgupta, and Pollack, without directly constructing $\alpha$. 

Explicit computation of Gross-Stark units over real quadratic base fields
December 19, 2012 7 / 20
Gross’ refined conjecture (1982)

Let $F$, $K$, $(p) = p\mathfrak{p}$, $S$, and $T$ be as above, and define the subgroup

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- This theorem was recently proved (Annals 2011), in the case described above, by Darmon, Dasgupta, and Pollack, without directly constructing $\alpha$.

- Brett and I give an algorithm using $G_{p,2}$ to compute the right side in $\mathbb{Q}_p$, and explicitly gives the irreducible polynomial $f_\alpha \in F[X]$ whose roots are the $\sigma(\alpha)$. 
The algorithm

First, explicitly choose a real quadratic field $F = \mathbb{Q}(\sqrt{d})$ with $d > 0$ to serve as base field; also choose a prime $(p)$ of $\mathbb{Z}$ that splits as $(p) = p\overline{p}$ in $\mathcal{O}_F$. Use characters on the ray class group $\mathcal{H} + (m)$ for a suitable ideal $m$ to specify a totally complex abelian extension $K$ of $F$ in which $p$ splits completely; the Galois group $G = \text{Gal}(K/F)$ is a specified subgroup of $\mathcal{H} + (m)$.

Compute the values $w_p \zeta_T(0; \sigma)$ and $\exp_p(-w_p \zeta'_S(0; \sigma))$ for all $\sigma \in G$; these are computed in $\mathbb{Q}_p$ using our $G_p, 2$ function, to any desired accuracy. By Gross' formula, we now know all $\sigma(\alpha)$ as elements of $\mathbb{Q}_p$ to any desired accuracy; we then must realize the coefficients of the polynomial $f_\alpha(x) = \prod_{\sigma \in G} (x - \sigma(\alpha))$ as elements of $F$.

Once we know the polynomial $f_\alpha \in F[x]$, then we know the field $K = F(\alpha)$ explicitly. At this point we verify computationally that $\alpha \in U_p$ and that $F(\alpha)$ is in fact the field $K$ originally specified; that is, it has the correct Galois group and the correct discriminant. And just like that, we have explicitly described an abelian extension $K$ of $F$, which was originally described algebraically, using $p$-adic analysis. Yeah baby!
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First, explicitly choose a real quadratic field $F = \mathbb{Q}(\sqrt{d})$ with $d > 0$ to serve as base field; also choose a prime $(p)$ of $\mathbb{Z}$ that splits as $(p) = pp$ in $O_F$.

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Computation of $\zeta'_{S,p}(0, \sigma)$

Using a continued fraction algorithm due to Hayes, each partial zeta function $\zeta_{S,p}(0, \sigma)$ can be decomposed into a finite sum of Shintani zeta functions

$$\zeta_S(0, \sigma) = \sum_{j=1}^{M} z_2(0, (\{z_j\}, \langle w_j \rangle, (\beta_j^{(1)}, \beta_j^{(2)}))).$$

Also holds, where $G_{p,2}(x; (\omega_1, \omega_2))$ is our $p$-adic double log gamma function.
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Using a formula due to Shintani, its partial derivative at \( s = 0 \) can be expressed as \( \zeta'_S(0, \sigma) \)

\[
= \sum_{j=1}^{M} \log \Gamma_2(\{z_j\} + \langle w_j \rangle \beta_j^{(1)}, (1, \beta_j^{(1)})) + \log \Gamma_2(\{z_j\} + \langle w_j \rangle \beta_j^{(2)}, (1, \beta_j^{(2)}))
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where \( \Gamma_2(x; (\omega_1, \omega_2)) \) is the complex double log gamma function.
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where $\Gamma_2(x; (\omega_1, \omega_2))$ is the complex double log gamma function.

- Using results of Kashio and T.-Y., the $p$-adic analogue $\zeta_{S,p}'(0, \sigma)$

$$= \sum_{j=1}^{M} G_{p,2}(\{z_j\} + \langle w_j \rangle (\beta_j)_p, (1, (\beta_j)_p)) + G_{p,2}(\{z_j\} + \langle w_j \rangle (\beta_j)_{\overline{p}}, (1, (\beta_j)_{\overline{p}}))$$

also holds, where $G_{p,2}(x; (\omega_1, \omega_2))$ is our $p$-adic double log gamma function.
The $p$-adic double log gamma function $G_{p,2}$

Initially defined on $\mathbb{C}_p \setminus \mathbb{Z}_p$ by a $p$-adic double integral, we actually compute these $G_{p,2}$ values by our “large $x$” expansion

$$G_{p,2}(x; \bar{\omega}) = -\frac{1}{2} B_{2,2}(x; \bar{\omega}) \log p x + \frac{3}{4 \omega_1 \omega_2} x^2 + B_{2,1}(0; \bar{\omega}) x$$

$$+ \sum_{j=3}^{\infty} \frac{(-1)^j B_{2,j}(0; \bar{\omega})}{j(j-1)(j-2)} x^{2-j},$$

which converges for $|x|_p > \max\{|\omega_1|_p, |\omega_2|_p\}$. Here the $B_{2,j}(x; \bar{\omega})$ are second-order Bernoulli polynomials.
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which converges for $|x|_p > \max\{|\omega_1|_p, |\omega_2|_p\}$. Here the $B_{2,j}(x; \bar{\omega})$ are second-order Bernoulli polynomials.

- If this series is truncated after the $j = m$ term, the approximation obtained for $G_{p,2}(x, (\omega_1, \omega_2))$ is accurate to at least $k$ $p$-adic digits, where

\[
k \geq \begin{cases} 
    m - 3 - \left\lfloor \frac{\log(m+1)}{\log p} \right\rfloor, & p > 2; \\
    m - 4 - \left\lfloor \frac{\log(m+1)}{\log p} \right\rfloor, & p = 2.
\end{cases}
\]
Realizing the coefficients in $F = \mathbb{Q}(\sqrt{d})$

So we can compute the $p$-adic expansions of the coefficients $\lambda_i$ of

$$f_\alpha(x) = \prod_{\sigma \in G} (x - \sigma(\alpha)) = x^n - \lambda_{n-1}x^{n-1} + \lambda_{n-2}x^{n-2} - \cdots + \lambda_0 \in F[x]$$

in $\mathbb{Q}_p$ to as many $p$-adic digits as we like. How do we realize them in $F = \mathbb{Q}(\sqrt{d})$?
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**Lemma.** If $\lambda_j = a_j + b_j \theta$ with $a_j, b_j \in \mathbb{Q}$, then $a_j, b_j$ are both of the form $cp^\nu$ where $\nu \in \mathbb{Z}$ are given in terms of the integers $\{w_p \zeta_T(0, \sigma)\}_{\sigma \in G}$, and

$$|b_j| \leq 2 \binom{n}{j} / \sqrt{d}$$

and

$$|a_j| \leq \begin{cases} \binom{n}{j}, & d \equiv 0 \text{ mod } (4), \\ \binom{n}{j}(1 + 1/\sqrt{d}), & d \equiv 1 \text{ mod } (4). \end{cases}$$
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**Here** $\{1, \theta\}$ is a $\mathbb{Z}$-basis for $\mathcal{O}_F$ satisfying $\theta_p - \theta_p^{-1} = \sqrt{d}$; that is,

$$\theta = \begin{cases} \sqrt{d}/2, & d \equiv 0 \text{ mod } (4), \\ (1 + \sqrt{d})/2, & d \equiv 1 \text{ mod } (4). \end{cases}$$
Realizing the coefficients, continued

The “trace” coefficient $\lambda_{n-1} = \sum_{\sigma \in G} \sigma(\alpha) \in F$ has $p$-adic absolute value $|\lambda_{n-1}|_p = p^r$, where $r = \max\{w_p \zeta_T(0, \sigma)\}_{\sigma \in G}$; but since $\alpha \in U_p$ it has $\bar{p}$-adic absolute value $|\lambda_{n-1}|_{\bar{p}} \leq 1$. 
The “trace” coefficient $\lambda_{n-1} = \sum_{\sigma \in G} \sigma(\alpha) \in F$ has $\mathfrak{p}$-adic absolute value $|\lambda_{n-1}|_\mathfrak{p} = \mathfrak{p}^r$, where $r = \max\{w_\mathfrak{p} \zeta_T(0, \sigma)\}_{\sigma \in G}$; but since $\alpha \in U_\mathfrak{p}$ it has $\mathfrak{p}$-adic absolute value $|\lambda_{n-1}|_\mathfrak{p} \leq 1$.

If $\lambda_{n-1} = (c_{n-1} + e_{n-1} \theta)/\mathfrak{p}^r$ with $c_{n-1}, e_{n-1} \in \mathbb{Z}$, and we obtain an approximation $\beta \in \mathbb{Z}_\mathfrak{p}$ to $\mathfrak{p}^r(\lambda_{n-1})_\mathfrak{p}$ accurate to $N$ digits, where $N \geq r$, then

$$|c_{n-1} + e_{n-1} \theta_\mathfrak{p} - \beta|_\mathfrak{p} \leq \mathfrak{p}^{-N} \quad \text{and} \quad |c_{n-1} + e_{n-1} \theta_\mathfrak{p}|_\mathfrak{p} \leq \mathfrak{p}^{-r}$$

$$\implies |c_{n-1} + e_{n-1} \theta_\mathfrak{p} - \beta - (c_{n-1} + e_{n-1} \theta_\mathfrak{p})|_\mathfrak{p} \leq \mathfrak{p}^{-r}$$

$$\implies |e_{n-1}(\theta_\mathfrak{p} - \theta_\mathfrak{p}) - \beta|_\mathfrak{p} = |e_{n-1} \sqrt{d} - \beta|_\mathfrak{p} \leq \mathfrak{p}^{-r}$$

$$\implies |e_{n-1} - \beta/\sqrt{d}|_\mathfrak{p} \leq \mathfrak{p}^{-r}$$

so $\beta/\sqrt{d}$ gives the integer $e_{n-1}$ accurate to at least $r$ base $p$ digits.
Realizing the coefficients, continued

The “trace” coefficient \( \lambda_{n-1} = \sum_{\sigma \in G} \sigma(\alpha) \in F \) has \( p \)-adic absolute value \(|\lambda_{n-1}|_p = p^r\), where \( r = \max\{w_p\zeta_T(0, \sigma)\}_{\sigma \in G} \); but since \( \alpha \in U_p \) it has \( \overline{p} \)-adic absolute value \(|\lambda_{n-1}|_{\overline{p}} \leq 1\).

- If \( \lambda_{n-1} = (c_{n-1} + e_{n-1}\theta)/p^r \) with \( c_{n-1}, e_{n-1} \in \mathbb{Z} \), and we obtain an approximation \( \beta \in \mathbb{Z}_p \) to \( p^r(\lambda_{n-1})_p \) accurate to \( N \) digits, where \( N \geq r \), then

\[
|c_{n-1} + e_{n-1}\theta_p - \beta|_p \leq p^{-N} \quad \text{and} \quad |c_{n-1} + e_{n-1}\theta_{\overline{p}}|_p \leq p^{-r}
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so \( \beta/\sqrt{d} \) gives the integer \( e_{n-1} \) accurate to at least \( r \) base \( p \) digits.

- This specifies the integer \( e_{n-1} \) to one of at most \( \lceil 4n/\sqrt{d} \rceil \) candidates; exactly one of these has \( c_{n-1} = \beta - e_{n-1}\theta_p \) recognizable as an integer.
Realizing the coefficients, continued

The “trace” coefficient $\lambda_{n-1} = \sum_{\sigma \in G} \sigma(\alpha) \in F$ has $p$-adic absolute value $|\lambda_{n-1}|_p = p^r$, where $r = \max\{w_p \zeta_T(0, \sigma)\}_{\sigma \in G}$; but since $\alpha \in U_p$ it has $\overline{p}$-adic absolute value $|\lambda_{n-1}|_{\overline{p}} \leq 1$.

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  $|c_{n-1} + e_{n-1}\theta_p - \beta|_p \leq p^{-N}$ and $|c_{n-1} + e_{n-1}\theta_{\overline{p}}|_p \leq p^{-r}$

  $\implies |c_{n-1} + e_{n-1}\theta_p - \beta - (c_{n-1} + e_{n-1}\theta_{\overline{p}})|_p \leq p^{-r}$

  $\implies |e_{n-1}(\theta_p - \theta_{\overline{p}}) - \beta|_p = |e_{n-1}\sqrt{d} - \beta|_p \leq p^{-r}$

  $\implies |e_{n-1} - \beta/\sqrt{d}|_p \leq p^{-r}$

so $\beta/\sqrt{d}$ gives the integer $e_{n-1}$ accurate to at least $r$ base $p$ digits.

- This specifies the integer $e_{n-1}$ to one of at most $\lceil 4n/\sqrt{d} \rceil$ candidates; exactly one of these has $c_{n-1} = \beta - e_{n-1}\theta_p$ recognizable as an integer.

- An analogous argument realizes the other coefficients $\lambda_i = (c_i + e_i\theta)/p^{r_i}$.
An example over the real quadratic field $F = \mathbb{Q}(\sqrt{29})$

The real quadratic field $F = \mathbb{Q}(\sqrt{29})$ is the splitting field over $\mathbb{Q}$ of the polynomial $f_{29}(x) = x^2 - x - 7$. The prime ideal $(7)$ of $\mathbb{Z}$ splits as $(7) = p\overline{p} = (6 + \sqrt{29})(6 - \sqrt{29})$ in $\mathcal{O}_F$; there are two embeddings of $F$ into $\mathbb{Q}_7$ corresponding to $p$ and $\overline{p}$; the two roots of $f_{29}$ are $\theta_p, \theta_{\overline{p}} = (1 \pm \sqrt{29})/2$. 
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- We set $T = \{p_\infty^{(1)}, p_\infty^{(2)}, q, \overline{p}\}$ and $m = q\overline{p}$, where $q$ is a prime ideal of $\mathcal{O}_F$ lying over $(13)$. The narrow ray class group $H_+(m)$ is isomorphic to $C_6 \times C_2$ and there is a sextic character $\chi$ on $H_+(m)$ with conductor $f(\chi) = mp_\infty^{(1)}p_\infty^{(2)}$; by class field theory there exists an abelian extension $K/F$ corresponding to the subgroup of characters generated by $\chi$ with $G = \text{Gal}(K/F)$ cyclic of order 6.
An example over the real quadratic field $F = \mathbb{Q}(\sqrt{29})$

The real quadratic field $F = \mathbb{Q}(\sqrt{29})$ is the splitting field over $\mathbb{Q}$ of the polynomial $f_{29}(x) = x^2 - x - 7$. The prime ideal $(7)$ of $\mathbb{Z}$ splits as $(7) = \mathfrak{p}\overline{\mathfrak{p}} = (6 + \sqrt{29})(6 - \sqrt{29})$ in $\mathcal{O}_F$; there are two embeddings of $F$ into $\mathbb{Q}_7$ corresponding to $\mathfrak{p}$ and $\overline{\mathfrak{p}}$; the two roots of $f_{29}$ are $\theta_{\mathfrak{p}}, \theta_{\overline{\mathfrak{p}}} = (1 \pm \sqrt{29})/2$.

- We set $T = \{\mathfrak{p}_\infty^{(1)}, \mathfrak{p}_\infty^{(2)}, q, \overline{p}\}$ and $m = q\overline{p}$, where $q$ is a prime ideal of $\mathcal{O}_F$ lying over $(13)$. The narrow ray class group $H_+(m)$ is isomorphic to $C_6 \times C_2$ and there is a sextic character $\chi$ on $H_+(m)$ with conductor $f(\chi) = mp_\infty^{(1)}p_\infty^{(2)}$; by class field theory there exists an abelian extension $K/F$ corresponding to the subgroup of characters generated by $\chi$ with $G = \text{Gal}(K/F)$ cyclic of order 6.
- By the form of the conductor $f(\chi)$ and the fact that $\chi(\mathfrak{p}) = 1$ we know that $K$ is totally complex, both $q$ and $\overline{p}$ ramify in the extension $K/F$, no other primes of $\mathcal{O}_F$ ramify, and $\mathfrak{p}$ splits completely in $K/F$. 

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- We set $T = \{p_1, p_\infty, q, \bar{p}\}$ and $m = q\bar{p}$, where $q$ is a prime ideal of $\mathcal{O}_F$ lying over $(13)$. The narrow ray class group $H_+(m)$ is isomorphic to $C_6 \times C_2$ and there is a sextic character $\chi$ on $H_+(m)$ with conductor $f(\chi) = mp_1p_\infty$; by class field theory there exists an abelian extension $K/F$ corresponding to the subgroup of characters generated by $\chi$ with $G = \text{Gal}(K/F)$ cyclic of order 6.

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- Our goal is to $7$-adically compute the six conjugates $\{\sigma(\alpha)\}_{\sigma \in G}$ of the Gross-Stark unit $\alpha \in K$ for the extension $K/F$ and the prime $p = 7$, and recognize the minimal polynomial $f_\alpha \in F[x]$, using only information from $F$. 
Choose an embedding of $K$ into $\mathbb{Q}_7$ corresponding to some prime ideal $\mathfrak{P}$ in $O_K$ lying above $p$ in $O_F$. 

Using a partial fraction algorithm we compute the values $\zeta_T(0,\sigma_0) = 0$ for the identity $\sigma_0$, and $\zeta_T(0,\sigma_2) = -2$, $\zeta_T(0,\sigma_3) = 0$, $\zeta_T(0,\sigma_4) = 2$, and $\zeta_T(0,\sigma_5) = 0$, where $\sigma$ is the generator of $G$ corresponding to $\chi$. 

The $p$-adic absolute values of the $\sigma(\alpha)$ are of the form $p^{-r}$ where $r = w_p \zeta_T(0,\sigma)$; since $w_7 = 6$ we have the $7$-adic absolute values \{7^{12}, 1, 1, 1, 1, 7^{-12}\} for $\alpha_\mathfrak{P}$ and its conjugates.

Recall that, since $\alpha \in U_p$, the absolute value of $\alpha$ with respect to every other absolute value on $K$ is 1.

The minimal polynomial of $\alpha$ over $F$ is of the form $f_\alpha(x) = x^6 - \lambda_5 x^5 + \lambda_4 x^4 - \lambda_3 x^3 + \lambda_2 x^2 - \lambda_1 x + 1 \in F[x]$ where each $\lambda_i = (c_i + e_i \theta) / 7^{12}$ for some $c_i, e_i \in \mathbb{Z}$, and $\theta = (1 + \sqrt{29})/2$.
Choose an embedding of $K$ into $\mathbb{Q}_7$ corresponding to some prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ lying above $p$ in $\mathcal{O}_F$.

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The $p$-adic absolute values of the $\sigma(\alpha)$ are of the form $p^{-r}$ where $r = \frac{\lambda}{7^{12}}$; since $w_7 = 6$ we have the 7-adic absolute values $\{7^{12}, 1, 1, 1, 1, 7^{-12}\}$ for $\alpha$ and its conjugates. 

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We can recognize all the $c_i, e_i$ by computing all the $\sigma(\alpha)_P$ accurate to just a few more than twelve 7-adic digits.
Absolute values of the Gross-Stark unit

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where each $\lambda_i = (c_i + e_i \theta)/7^{12}$ for some $c_i, e_i \in \mathbb{Z}$, and $\theta = (1 + \sqrt{29})/2$. 


Choose an embedding of $K$ into $\mathbb{Q}_7$ corresponding to some prime ideal $\mathfrak{P}$ in $\mathcal{O}_K$ lying above $p$ in $\mathcal{O}_F$.

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Computation of the values $\zeta'_{S,7}(0; \sigma)$

Gross’ formula states that $(\sigma(\alpha))_{\mathfrak{p}_3} = p^{w_p \zeta_T(0; \sigma)} \exp_p(-w_p \zeta'_{S,p}(0; \sigma))$ for all $\sigma \in G$.

We will compute the six values $(\sigma(\alpha))_{\mathfrak{p}_3} = 7^{6\zeta_T(0; \sigma)} \exp_7(-6 \zeta'_{S,7}(0; \sigma))$ in $\mathbb{Q}_7$. It is not obvious that $-6 \zeta'_{S,7}(0; \sigma)$ lies in the domain $\mathbb{Z}_7$ of $\exp_7$. Especially since we compute these values as sums of 7-adically large values.

It is remarkable that the values $\exp_7(-6 \zeta'_{S,7}(0; \sigma))$ should be algebraic. We compute these values in $\mathbb{Q}_7$. But Gross’ assertion that $\alpha \in U_{(p, t)}$ tells us the absolute values of the $\sigma(\alpha)$ with respect to every other embedding of $K$ into $\mathbb{C}$ or a $p$-adic field.

So not only are these values computable in $\mathbb{Q}_7$, and algebraic, but they are recognizable as specific algebraic numbers. Not only that, but they are special algebraic numbers – they generate a specific abelian extension of $F = \mathbb{Q}(\sqrt{29})$. 

Paul Thomas Young (College of Charleston)
Computation of the values $\zeta'_{S,7}(0; \sigma)$

Gross’ formula states that $(\sigma(\alpha))_{\wp} = p^{w_p\zeta_T(0;\sigma)} \exp_p(-w_p\zeta'_{S,p}(0;\sigma))$ for all $\sigma \in G$.

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Computation of the values $\zeta_{S,7}'(0; \sigma)$

Gross’ formula states that $(\sigma(\alpha))_\mathfrak{p} = p^{w_p \zeta_T(0;\sigma)} \exp_p(-w_p \zeta_{S,7}'(0;\sigma))$ for all $\sigma \in G$.

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Gross’ formula states that \((\sigma(\alpha))_p = \wp^w_p \zeta_T(0; \sigma) \exp_p(-\wp \zeta'_S, p(0; \sigma))\) for all \(\sigma \in G\). We will compute the six values \((\sigma(\alpha))_p = 7^6 \zeta_T(0; \sigma) \exp_7(-6 \zeta'_S, 7(0; \sigma))\) in \(\mathbb{Q}_7\).

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Computation of the values $\zeta_{S,7}'(0; \sigma)$

Each $\zeta_{S,7}'(0; \sigma)$ is a finite sum of $\zeta_{m\mathfrak{p},7}'(0; C_+)$ values over all ideal classes $C_+$ in a coset of a subgroup of the narrow ray class group $H_+(m\mathfrak{p})$. 

We have programmed all of this in PARI routines.
Computation of the values $\zeta'_{S,7}(0; \sigma)$

Each $\zeta'_{S,7}(0; \sigma)$ is a finite sum of $\zeta'_{mp,7}(0; C_+)$ values over all ideal classes $C_+$ in a coset of a subgroup of the narrow ray class group $H_+(mp)$.

- The partial fraction algorithm produces an ordered sequence of $\mathbb{Z}$-bases for an ideal $mc$ of $\mathcal{O}_F$ in the class $C_+$. 

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- This decomposes each term $\zeta'_{m\mathfrak{p},7}(0; C_+)$ into a finite sum of derivatives of Shintani zeta functions, which we compute as a finite sum of values $G_{7,2}(x_i; \bar{\omega}_i)$, where the $x_i$ and $\bar{\omega}_i$ are given in terms of the parameters of the bases, and all satisfy $|x_i|_7 > ||\bar{\omega}_i||_7$. 

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- This means we can compute each of these terms in $\mathbb{Q}_7$ using our “large $x$” expansion

$$G_{7,2}(x; \bar{\omega}) = -\frac{1}{2}B_{2,2}(x; \bar{\omega}) \log_7 x + \frac{3}{4\omega_1\omega_2} x^2 + B_{2,1}(0; \bar{\omega}) x$$

$$+ \sum_{j=3}^{\infty} \frac{(-1)^j B_{2,j}(0; \bar{\omega})}{j(j-1)(j-2)} x^{2-j}.$$
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- We have programmed all of this in PARI routines.
Realizing the trace coefficient $\lambda_5$

By this method we compute the 7-adic approximation

$$\beta = 7^{12} \cdot \sum_{\sigma \in G} 7^{6\zeta_T(0,\sigma)} \cdot \exp_7(-6\zeta_{S,7}(0, \sigma))$$

$$= 1 + 3 \cdot 7 + 3 \cdot 7^2 + 7^3 + 4 \cdot 7^4 + 7^5 + 3 \cdot 7^7 + 3 \cdot 7^8$$
$$+ 6 \cdot 7^9 + 6 \cdot 7^{10} + 0 \cdot 7^{11} + O(7^{12}) = (133141033660\ldots)_7$$

to $7^{12} \lambda_5 = c_5 + e_5 \theta_p$, which in turn yields the approximation

$$\beta/\sqrt{29} = (113016104651\ldots)_7$$

to the integer $e_5$. 

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- We truncate this 7-adic expansion mod $7^{12}$ as indicated, giving the integer $e = 3655104881$; considering the bound $2\binom{6}{5}/\sqrt{29} = 2.2283...$, we know that $e_5$ must be exactly one of $\{e - 2 \cdot 7^{12}, e - 7^{12}, e, e + 7^{12}\}$. 
Realizing the trace coefficient $\lambda_5$

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- Exactly one of these choices for $e_5$ should be such that $\beta - e_5 \theta_p$ is recognizable as an integer $c_5$ satisfying the bound $|c_5| \leq (6)_5(1 + 1/\sqrt{29}) \cdot 7^{12}$. 
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By this method we compute the 7-adic approximation

$$\beta = 7^{12} \cdot \sum_{\sigma \in G} 7^{6\zeta_T(0,\sigma)} \cdot \exp_7(-6\zeta_S,7(0,\sigma))$$

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- Exactly one of these choices for $e_5$ should be such that $\beta - e_5 \theta_p$ is recognizable as an integer $c_5$ satisfying the bound $|c_5| \leq \binom{6}{5}(1 + 1/\sqrt{29}) \cdot 7^{12}$.

- Beyond the $7^{12}$s digit, such an integer must have 7-adic digits either all zeros or all sixes.
Realizing the trace coefficient $\lambda_5$

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- We find that $e_5 = e - 7^{12} = -10186182320$ and $c_5 = -849169895$. 

Paul Thomas Young (College of Charleston) 
Explicit computation of Gross-Stark units over real quadratic fields
The Gross-Stark unit $\alpha$ given explicitly

The same method determines the other coefficients $\lambda_i$, which are also symmetric functions of the $\sigma(\alpha)$. The minimal polynomial satisfied by the Gross-Stark unit $\alpha$ over $F$ is

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where $\theta = (1 + \sqrt{29})/2$ is a root of the polynomial $x^2 - x - 7$ such that $\{1, \theta\}$ is a basis for $\mathcal{O}_F$ over $\mathbb{Z}$. We verified this numerically to sixty-seven 7-adic digits.
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- In the spirit of Hilbert’s Twelfth Problem, we have given a 7-adic analytic construction of a specific totally complex abelian extension $K$ of $F$, using only information from $F$. 
Problems for the next century

So we can calculate the Gross-Stark units $\alpha$ attached to relative abelian totally complex extensions of real quadratic fields; now what?

Improve the implementation in PARI; find more efficient ways to choose the prime $(p) = p$. We found some new relations in the course of this work; for example, for each class $C^+ \in H^+(mp)$ we have $\zeta'(mp)_p(0, [\nu] + C^+)$, where $\nu := N(mp) - 1$ and $[\nu]$ denotes the narrow class modulo $mp$ to which the principal ideal $(\nu)$ belongs; are there others? Can we exploit them?

Can we use our explicit analytic construction of the Gross-Stark units to suggest an algebraic description of them? For example, in the case where the base field is $F = \mathbb{Q}$ the Gross-Stark units are roots of unity or sums thereof.

Can a more constructive independent proof of Darmon-Dasgupta-Pollack theorem be given?

Extend the algorithm to higher-degree totally real base fields $F$, using higher $p$-adic multiple log gamma functions we have developed. (note: the Gross-Stark conjecture is only known conditionally in the general case.)
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