Apollonian Circle Packings

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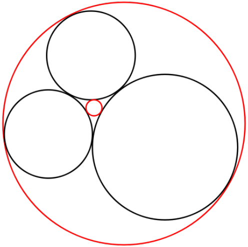
Apollonius's Theorem

Theorem

Given three mutually tangent circles, there exists exactly two circles tangent to all three.

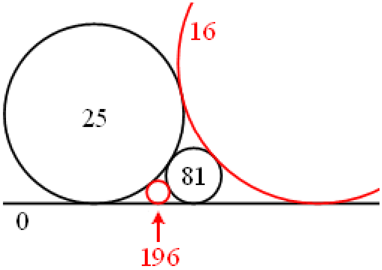


Descartes' Theorem

Theorem

Given four mutually tangent circles, the curvatures a, b, c, and d satisfy the Descartes equation

$$a^2 + b^2 + c^2 + d^2 = \frac{1}{2}(a + b + c + d)^2.$$



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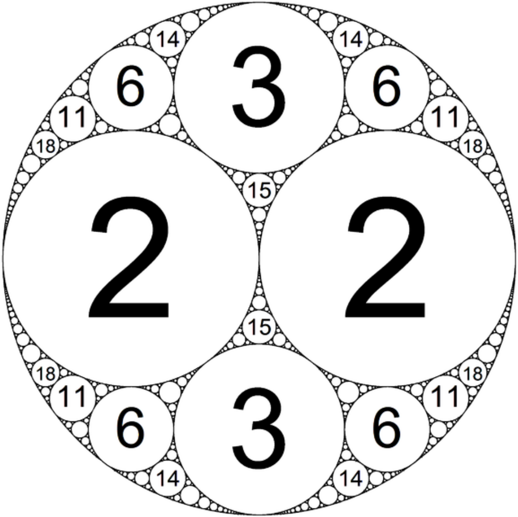

$(-1, 2, 2, 3)$ Apollonian Circle Packing

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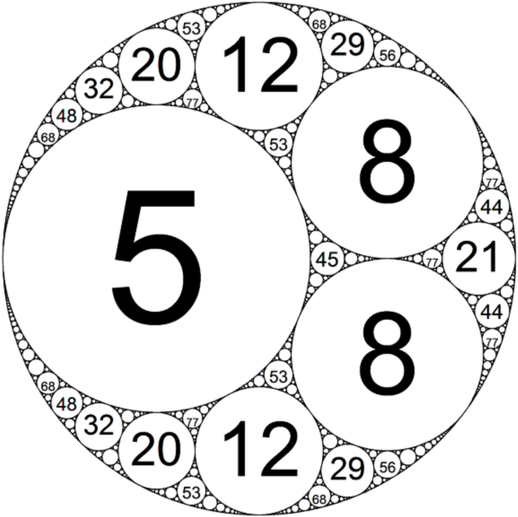

$(-3, 5, 8, 8)$ Apollonian Circle Packing

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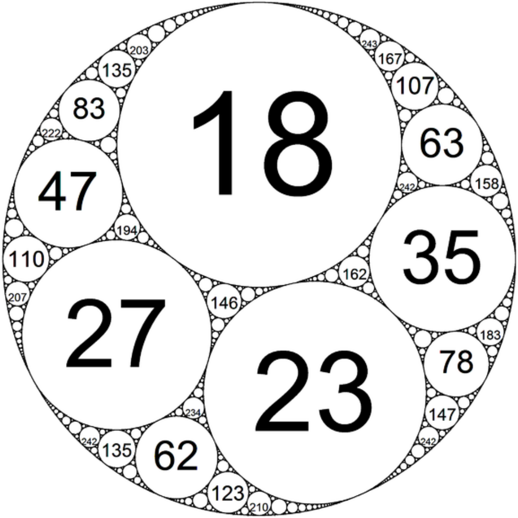

$(-10, 18, 23, 27)$ Apollonian Circle Packing

If (a, b, c, d) is a Descartes quadruple, then so is (a, b, c, d') where

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Corollary

If an Apollonian circle packing has four mutually tangent circles with integral curvatures, then all curvatures will be integral.

Root Quadruples

A *root quadruple* is a Descartes quadruple (a, b, c, d) with $a \leq 0 \leq b \leq c \leq d$ and $a + b + c \geq d$.

$(-1, 2, 2, 3) \qquad (-2, 3, 6, 7) \quad (-3, 4, 12, 13) \quad (-3, 5, 8, 8)$

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$(-4, 5, 20, 21)$	$(-4, 8, 9, 9)$	$(-5, 6, 30, 31)$	$(-5, 7, 18, 18)$

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Root quadruples correspond to the curvatures of the four largest circles (taking the negative to be the curvature of the outside circle). Root quadruples are unique, although an Apollonian circle packing may contain more than one Descartes configuration with these curvatures.

Number of Root Quadruples

Question: How many distinct packings are there?

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Infinitely many: $(-n, n+1, n(n+1), n(n+1)+1)$.

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Better Question: How many distinct packings are there with outside circle having curvature $-n$?

Root Quadruples

How many primitive root quadruples with negative element $-n$?

n	1	2	3	4	5	6	7	8	9	10
$N(-n)$	1	1	2	2	2	3	3	3	4	4
n	11	12	13	14	15	16	17	18	19	20
$N(-n)$	4	6	4	5	6	5	5	7	6	6

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n	1009	1013	2003	2011	3001	3011	4001	4003
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It would appear that $N(-p) \approx \frac{p}{4}$ for large prime p.

Characterization of Root Quadruples

Theorem (Graham, Lagarias, Mallows, Wilks, Yan)

The Apollonian root quadruples $(-n, x, y, z)$ *are in one-to-one correspondence with positive definite integral binary quadratic forms of discriminant* $-4n^2$ *having non-negative middle coefficient.*

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Example: The associated BQF of $(-6, 10, 15, 19)$ is $[4, 0, 9]$.

Characterization of Root Quadruples

$$[A, B, C] = [-n + x, -n + x + y - z, -n + y]$$

Primitive root quadruples correspond to reduced binary quadratic forms having nonnegative middle coefficient. The number $N(-n)$ of primitive root quadruples with least element $-n$ satisfies

$$N(-n) = h^{\pm}(-4n^2)$$

where $h^{\pm}(-4n^2)$ is the number of equivalence classes of positive definite primitive binary integral forms of discriminant $-4n^2$.

Using analytic class number formulas, there is an exact formula for $N(-n)$ depending only on the prime factorization of n.

Open Question (GLMWY)

Is there a direct way for determining the root quadruple to which a given Descartes quadruple belongs?

Apollonian Group

If (a, b, c, d) is a Descartes quadruple, then so is (a, b, c, d') where

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Apollonian Group

If (a, b, c, d) is a Descartes quadruple, then so is (a, b, c, d') where

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$$S_1 = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{bmatrix}$$

The *Apollonian group* $\mathcal{A}$ is the subgroup of $\mathrm{GL}_4(\mathbb{Z})$ generated by S_1, S_2, S_3, and S_4. (Note: $S_i^2 = I$.)

Properties of the Apollonian Group

Descartes Quadratic Form:

$$\mathcal{Q}(x_1, x_2, x_3, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2$$

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Orthogonal Group:

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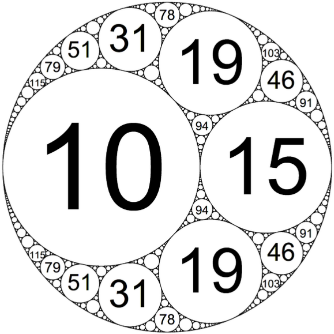
Let $\mathcal{A}$ be the Apollonian group and $\mathcal{Q}$ be the Decartes quadratic form. Then

- $\mathcal{A}$ is "small": $\mathcal{A}$ is an infinite-index subgroup of the orthogonal $O_{\mathcal{Q}}(\mathbb{Z})$ fixing $\mathcal{Q}$.
- $\mathcal{A}$ is "not too small": $\mathcal{A}$ is Zariski dense in $O_{\mathcal{Q}}(\mathbb{Z})$.

Apollonian Group Action

The Apollonian group $\mathcal{A}$ action on integral Descartes quadruples allows one to "walk around" a given Apollonian circle packing.



There will be $4 \cdot 3^{n-2}$ circles at the n^{th} generation.

Reduction Algorithm (GLMWY)

Input:
A Descartes quadruple (a, b, c, d) with $a + b + c + d > 0$.

1. Test in order $1 \leq i \leq 4$ whether some S_i decreases the sum $a + b + c + d$. If so, apply it to produce a new quadruple and continue.

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2. If no S_i decreases the sum, order the elements of the quadruple in increasing order and halt.

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If a, b, c, d are integers, then the reduction algorithm will halt at a root quadruple in finitely many steps.

Associated Quadratic Form

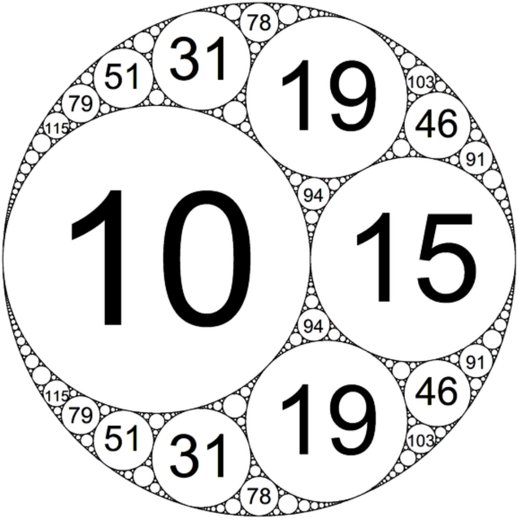
For an ordered Descartes quadruple $\mathbf{v} = (a, b, c, d)^T$ $(a \leq b \leq c \leq d)$, define the associated quadratic form $Q(X, Y) = AX^2 + BXY + CU^2$ to be

$$[A, B, C] = [a + b, a + b + c - d, a + c].$$

The discriminant is $-4a^2$.

Theorem (Morgenstern, E.)

The associated quadratic form for an ordered Descartes quadruple $\mathbf{v} = (a, b, c, d)^T$ *is (improperly) equivalent to the associated quadratic forms of* $S_2\mathbf{v}$, $S_3\mathbf{v}$, *and* $S_4\mathbf{v}$. *In particular, the discriminant is invariant under* $B_1 = \langle S_2, S_3, S_4 \rangle$.



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Theorem (Morgenstern, E.)

If $\mathbf{v} = [a, b, c, d]$ is an ordered Descartes quadruple with associated quadratic form $[A, B, C]$, then the reduced form $[A', B', C']$ will produce another (usually) ordered Descartes quadruple $\mathbf{v}'$ in the same Apollonian packing as $\mathbf{v}$:

$$\mathbf{v}' = (a', b', c', d') = (A' - a, C' - a, A' + B' + C' - a, a).$$

Unless $\mathbf{v}$ is a root quadruple, $a' + b' + c' + d' < a + b + c + d$. Furthermore, the associated quadratic form of $\mathbf{v}'$ will have smaller discriminant than the associated quadratic form of $\mathbf{v}$.

Improved Reduction Algorithm

Input:
A Descartes quadruple (a, b, c, d) with $a + b + c + d > 0$.

1. Find the associated quadratic form (A, B, C).

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3. Produce the (usually) ordered quadruple

$$(a', b', c', d') = (A' - a, C' - a, A' + B' + C' - a, a)$$

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4. Repeat steps 1-3, inputing the result of the previous iteration, until the smallest term is nonpositive.

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4. Repeat steps 1-3, inputing the result of the previous iteration, until the smallest term is nonpositive.
5. When $a \leq 0$, perform steps 1-3 once more, then return the root quadruple $(-n, x, y, z)$.

Example

Descartes Quadruple:

$$(a, b, c, d) = (1200139418, 3576170042, 9994848203, 29197576575)$$

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$$[A', B', C'] = [1208222152, -1110731620, 1447387637]$$
$$(a, b, c, d) = (8082734, 247248219, 344738751, 1200139418)$$
$$[A', B', C'] = [8082724, -7584880, 9862169]$$
$$(a, b, c, d) = (-10, 1779435, 2277279, 8082734)$$
$$[A', B', C'] = [4, 0, 25]$$

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Root quadruple:

$$(a, b, c, d) = (-10, 14, 35, 39)$$

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Proof. To be proven or disproven by Rob at next year's WCNT.