Der Wissenschaftsfonds.

Class Towers and Capitulation over Quadratic Fields

Conference:	West Coast Number Theory 2013
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Date:	December 15 – 19, 2013
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	M. F. Newman (ANU, Canberra)
Dedication:	to the memory of O. Taussky-Todd
	(∗ 1906 – † 1995)

§ 0. Summary of Aims

Section 1. **Capitulation**

Section 2. **Distribution of Second Class Groups**

§§ 1–2 are skipped almost entirely, since the presentation is limited with 15 minutes. They will, however, be contained in the official PDF document `http://www.algebra.at/Asilomar2013.pdf`.

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3.2. On the one hand, to underpin the caveats of **Brink/Gold** [3], who had doubts about Scholz/Taussky's claim, but on the other hand, to show that the arguments given by Brink/Gold are unable to invalidate the Scholz/Taussky claim.

§ 1. Kernels and Targets of Artin Transfers

Definition 1.1.
$p \geq 2$ a prime number,
G a pro-p group of generator rank $d(G) = 2$,
$H_1, \ldots, H_{p+1} \lhd G$ its maximal subgroups,
$T_i = T_{G,H_i} : G/G' \to H_i/H_i'$, $gG' \mapsto$

$$T_i(gG') = \begin{cases} g^p H_i' & \text{if } g \in G \setminus H_i, \\ g^{1+t+\cdots+t^{p-1}} H_i' & \text{if } g \in H_i, \end{cases}$$

for any $t \in G \setminus H_i$ and $1 \leq i \leq p+1$,
the *Artin transfers* from G to the H_i [1].

[1] E. Artin, Idealklassen in Oberkörpern und allgemeines Reziprozitätsgesetz, *Abh. Math. Sem. Univ. Hamburg* **7** (1929), 46–51.

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The family $\varkappa(G) = (\ker(T_i))_{1 \leq i \leq p+1}$
is called the *transfer kernel type* (TKT) of G
[T2], [T4].

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§ 2. Capitulation of p-Classes

Definition 2.1.

K a number field of p-class rank $r_p(K) = 2$,
$L_1, \ldots, L_{p+1}$
its unramified cyclic extension fields of degree p,
$j_i = j_{L_i|K} : \mathrm{Cl}_p(K) \to \mathrm{Cl}_p(L_i)$
the extension homomorphisms of p-classes.

The family $\varkappa(K) = (\ker(j_i))_{1 \le i \le p+1}$
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is called the *p-class group type* of K [T4].

Theorem 2.1. (Artin, 1929 [1])
The p-capitulation type $\varkappa(K)$, resp. p-class group type $\tau(K)$, of K coincides with the TKT $\varkappa(G)$, resp. TTT $\tau(G)$, of the nth p-class group $G = \mathrm{G}_p^n(K)$, for any $2 \le n \le \infty$.

$$\begin{array}{ccccc} & & j_{L_i|K} & & \\ & \mathrm{Cl}_p(K) & \longrightarrow & \mathrm{Cl}_p(L_i) & \\ \text{Artin} & \downarrow & & \downarrow & \text{Artin} \\ \text{isomorphism} & G/G' & \longrightarrow & H_i/H_i' & \text{isomorphism} \\ & & T_{G,H_i} & & \end{array}$$

§ 3. Exact Length of 3-Class Towers

Theorem 3.1. (Scholz & Taussky, 1934 [8])
The Galois group $G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the second Hilbert 3-class field over the complex quadratic field $K = \mathbb{Q}(\sqrt{-9748})$ has transfer kernel type E

$$\varkappa(G) = (2,3,3,4) \sim (2,4,3,4)$$

and the 3-class numbers of the non-Galois absolute cubic subfields $K_1,\ldots,K_4$ of the unramified cyclic cubic extension fields $L_1,\ldots,L_4$ of K are given by

$$(\mathrm{h}_3(K_i))_{1\le i\le 4} = (9,3,3,3)\,.$$

[8] A. Scholz und O. Taussky, Die Hauptideale der kubischen Klassenkörper imaginär quadratischer Zahlkörper: ihre rechnerische Bestimmung und ihr Einfluß auf den Klassenkörperturm, *J. Reine Angew. Math.* **171** (1934), 19–41.

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Corollary 3.1. (Mayer, 2009 [T3])
The Galois group $G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the second Hilbert 3-class field over the complex quadratic field $K = \mathbb{Q}(\sqrt{-9748})$ has transfer target type

$$\tau(G) = [(9,27),(3,9)^3].$$

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Definition 3.1. For a finite metabelian p-group $G = \langle x, y \rangle$ with generator rank $d(G) = 2$ and main commutator $s_2 = [y, x]$, the ideal

$$\mathfrak{A}(G) = \{ f(X,Y) \in \mathbb{Z}[X,Y] \mid s_2^{f(x-1,y-1)} = 1 \}$$

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Theorem 3.2. (Scholz & Taussky, 1934 [8]) The annihilator $\mathfrak{A}(G)$ of the Galois group $G = \mathrm{Gal}(\mathrm{F}_3^2(K)|K)$ of the second Hilbert 3-class field over any quadratic field $K = \mathbb{Q}(\sqrt{D})$ with transfer kernel type E

$$\varkappa(G) = (2,3,3,4) \sim (2,4,3,4)$$

is one of the ideals

$$\mathfrak{X}_\alpha = \langle X^\alpha, XY, Y^2, X^2 + 3X + 3 \rangle$$

with even $\alpha \geq 4$.

A Deep Mystery since 80 Years

Claim 3.1. (Scholz & Taussky, 1934 [8])
The 3-class field tower over the complex quadratic field $K = \mathbb{Q}(\sqrt{-9748})$ terminates at the second stage,

$$\mathrm{F}_3^3(K) = \mathrm{F}_3^2(K),$$

resp. has length $\ell_3(K) = 2$.

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Claim 3.2. (Heider & Schmithals, 1981 [5])
The 3-class field tower over any complex quadratic field $K = \mathbb{Q}(\sqrt{D})$ with 3-capitulation type E

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[5] F.-P. Heider und B. Schmithals, Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen, *J. Reine Angew. Math.* **336** (1982), 1–25.

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Claim 3.2 on p. 20 of Heider and Schmithals [5] has been used in the table on p. 84 of our paper [7], where the rows Nr. 6, 8, 9, and 14 are marked by the symbol $\times$ to indicate a two-stage tower.

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A Caveat by Brink and Gold

Theorem 3.3. (Brink and Gold, 1987 [2,3])
The 3-groups with parametrized presentation
$M_2(\beta) = \langle x, y, s_2, s_3, t_3, u \mid$
$[y, x] = s_2, [s_2, x] = s_3, [s_2, y] = t_3,$
$[s_3, x] = s_2^{-3} s_3^{-3} t_3^6,\ [s_3, y] = u^2,\ [s_3, s_2] = u,$
$[t_3, x] = [t_3, y] = [t_3, s_2] = [t_3, s_3] = 1,\ t_3^3 = u,$
$x^3 = t_3^{-1},\ y^3 = s_2^{-3} s_3^{-1},\ s_2^{3^\beta} = s_3^{3^\beta} = u^3 = 1 \rangle$
have cyclic second derived subgroup $M_2(\beta)''$ of order 3, for all parameter values $\beta \geq 2$. Hence, they are non-metabelian with derived length

$$\mathrm{dl}(M_2(\beta)) = 3.$$

The annihilator ideal $\mathfrak{A}(G)$ of the metabelianization $G = M_2(\beta)/M_2(\beta)''$ is given by

$$\mathfrak{X}_\alpha = \langle X^\alpha, XY, Y^2, X^2 + 3X + 3 \rangle$$

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Claim 3.3. (Brink and Gold, 1987 [2,3])
The groups $M_2(\beta)$ with $\beta \geq 2$ are possible candidates for Galois groups $\mathrm{Gal}(M|K)$ of unramified cyclic cubic extensions $M|\mathrm{F}_3^2(K)$ within the third Hilbert 3-class field $\mathrm{F}_3^3(K)$ over complex quadratic fields $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, with 3-capitulation type E

$$\varkappa(K) = (2,3,3,4) \sim (2,4,3,4).$$

Crucial Ingredients for the Disproof

Definition 3.2. $p \geq 3$ an odd prime.
A pro-p group G is called a σ-*group*,
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For any *quadratic* field $K = \mathbb{Q}(\sqrt{D})$,
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$p \geq 3$ an odd prime,
G a pro-p group,
$d(G) = \dim_{\mathbb{F}_p}(\mathrm{H}^1(G, \mathbb{F}_p))$ the *generator rank* of G,
$r(G) = \dim_{\mathbb{F}_p}(\mathrm{H}^2(G, \mathbb{F}_p))$ the *relation rank* of G.

Definition 3.3. A pro-p group G which satisfies the equation $r(G) = d(G)$ is said to have a *balanced presentation*, or to be a *Schur group*.

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Theorem 3.5. (Shafarevich, 1963 [10])
The p-tower group $\mathrm{G}_p^{\infty}(K)$ of a *complex quadratic* field $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, is a Schur group.

Theorem 3.6. (Mayer, Boston & Bush, 2012) There are exactly two non-isomorphic metabelian 3-groups G_1 and G_2 with transfer kernel type E

$$\varkappa(G_i) = (2, 3, 3, 4) \sim (2, 4, 3, 4)$$

and transfer target type

$$\tau(G_i) = [(9, 27), (3, 9)^3].$$

G_1 and G_2 do not have a balanced presentation. Further, there are exactly two non-isomorphic non-metabelian 3-groups H_1 and H_2 such that $G_i \simeq H_i/H_i''$. H_1 and H_2 are Schur σ-groups of derived length $\mathrm{dl}(H_i) = 3$.

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Remark 3.1. The identifiers of these 3-groups are

$$G_1 \simeq \langle 2187, 302\rangle,$$
$$G_2 \simeq \langle 2187, 306\rangle$$

in the SmallGroups library, resp.

$$H_1 \simeq \langle 729, 54\rangle - \#2; 2,$$
$$H_2 \simeq \langle 729, 54\rangle - \#2; 6$$

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Corollary 3.6. (Mayer, Boston & Bush, 2012) The 3-class field tower over the complex quadratic field $\mathbb{Q}(\sqrt{-9748})$ terminates at the third stage,

$$\mathrm{F}_3^4(K) = \mathrm{F}_3^3(K) > \mathrm{F}_3^2(K),$$

resp. has exact length $\ell_3(K) = 3$.

Brink and Gold — Tidy !!!

Theorem 3.7. (Mayer and Newman, 2013)
Brink and Gold's 3-groups $G = M_2(\beta)$ with parameter values $\beta \geq 2$ are of order $3^{2\beta+4}$, class $2\beta + 1$, and fixed coclass 3.

None of these groups has a balanced presentation and further they are all of transfer kernel type c

$$\varkappa(G) = (2, 0, 3, 4).$$

Their metabelianizations $M_2(\beta)/M_2(\beta)''$ are the mainline groups of order $3^{2\beta+3}$ on the tree $\mathcal{T}_2^*(\langle 243, 8\rangle)$.

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Corollary 3.7. (Mayer and Newman, 2013)
None of Brink and Gold's 3-groups $M_2(\beta)$, $\beta \geq 2$, can be the Galois group $\mathrm{Gal}(M|K)$ of an unramified cyclic cubic extension $M|\mathrm{F}_3^2(K)$ within the third Hilbert 3-class field $\mathrm{F}_3^3(K)$ over any complex quadratic field $K = \mathbb{Q}(\sqrt{D})$, $D < 0$, with 3-capitulation type E

$$\varkappa(K) = (2, 3, 3, 4) \sim (2, 4, 3, 4).$$

FIGURE 1. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 8\rangle)$ restricted to σ-groups with balanced covers in ovals, Brink/Gold's groups in rectangles, projections to the metabelianizations, and formal identifiers

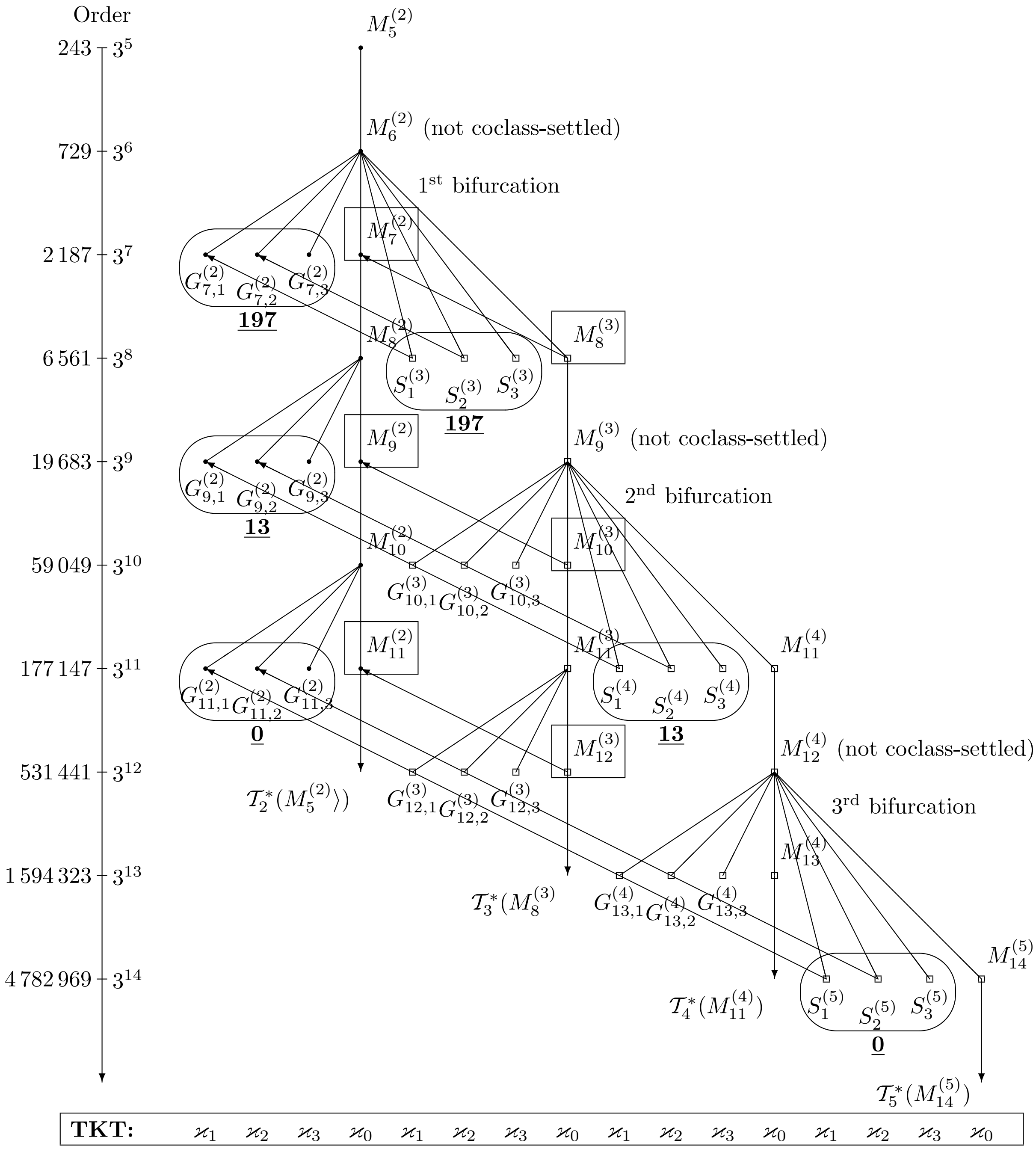


Figure 2. Normal lattice, including upper and lower central series, of a **three-stage** non-metabelian Schur σ-group G, e.g. $G = S_1^{(3)}$, with TKT E, class 5.

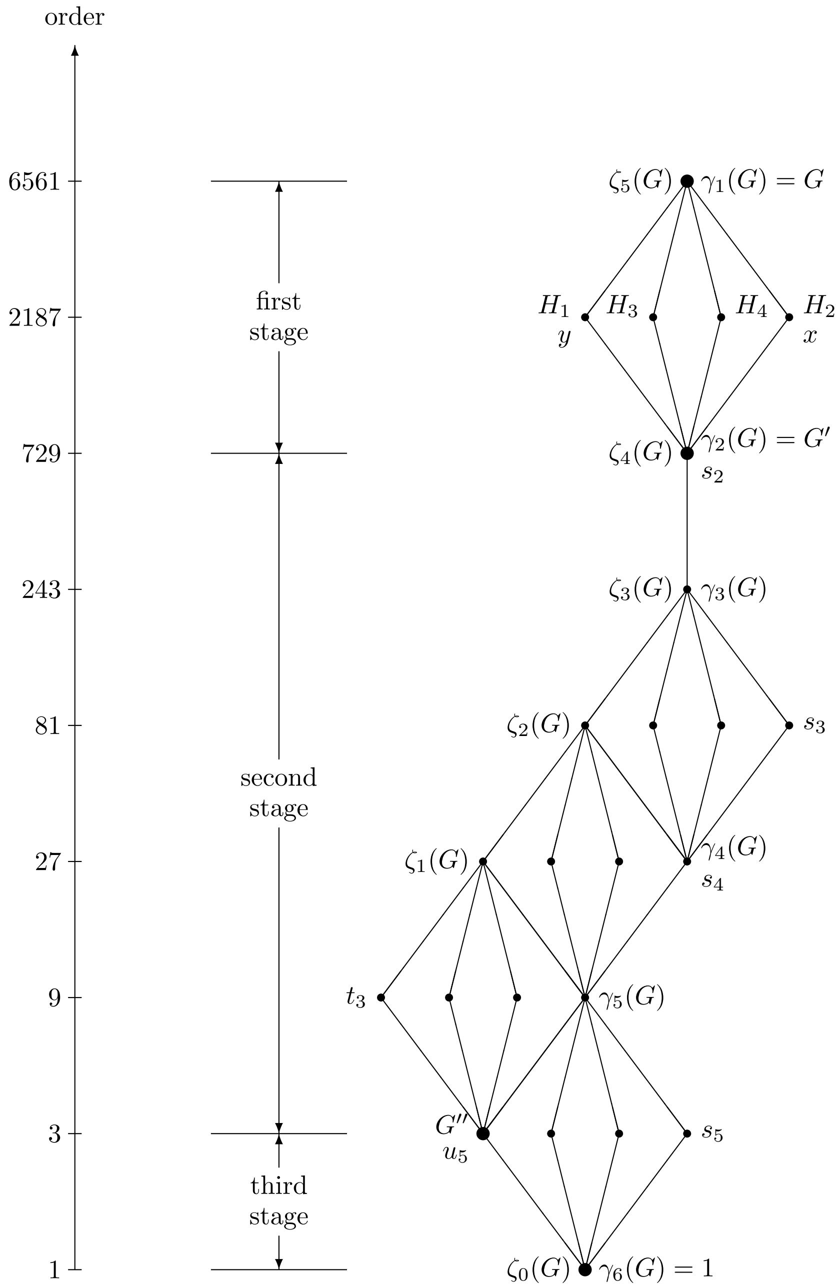
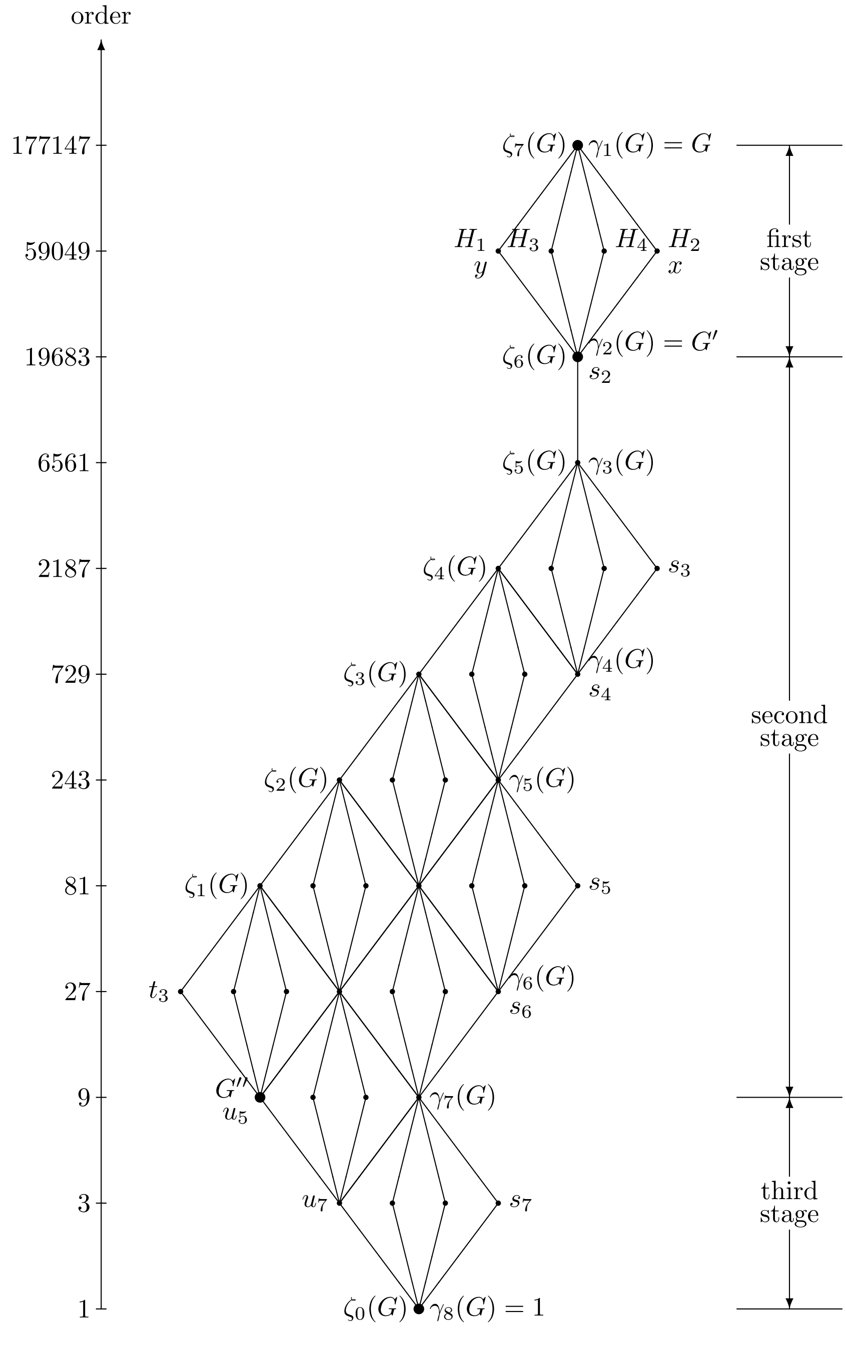


FIGURE 3. Normal lattice, including upper and lower central series, of a **three-stage** non-metabelian Schur σ-group G, e.g. $G = S_1^{(4)}$, with TKT E, class 7.

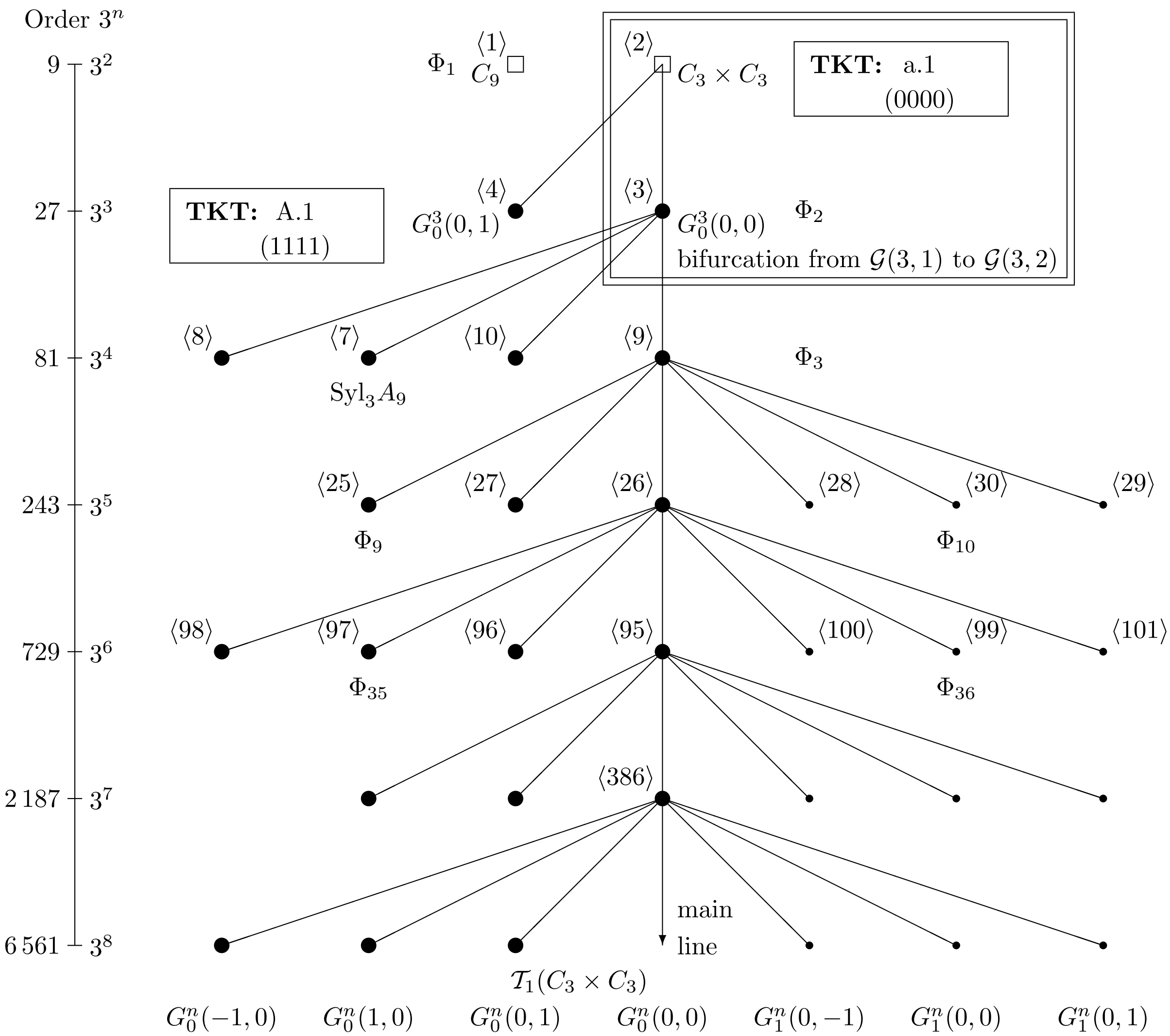


§ 4. Proof of Theorem 3.6

§ 4.1. Starting Generation of 3-Groups

We start our search for 3-groups with TKT in section E at the abelian root $C_3 \times C_3 \simeq \langle 9, 2\rangle$ of the unique coclass tree $\mathcal{T}_1$ in coclass graph $\mathcal{G}(3, 1)$. However, we leave this graph very quickly, since all 3-groups of maximal class have TKTs in sections a,A. The immediate descendant $G_0^3(0, 0) \simeq \langle 27, 3\rangle$ gives rise to a bifurcation from $\mathcal{G}(3, 1)$ to $\mathcal{G}(3, 2)$, but the following mainline vertex $G_0^4(0, 0) \simeq \langle 81, 9\rangle$ is coclass-settled and no further bifurcations can occur.

FIGURE 4. Starting 3-group generation at the top of coclass graph $\mathcal{G}(3,1)$



TKT:	a.3	a.3	a.2	a.1	a.1	a.1	a.1
	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

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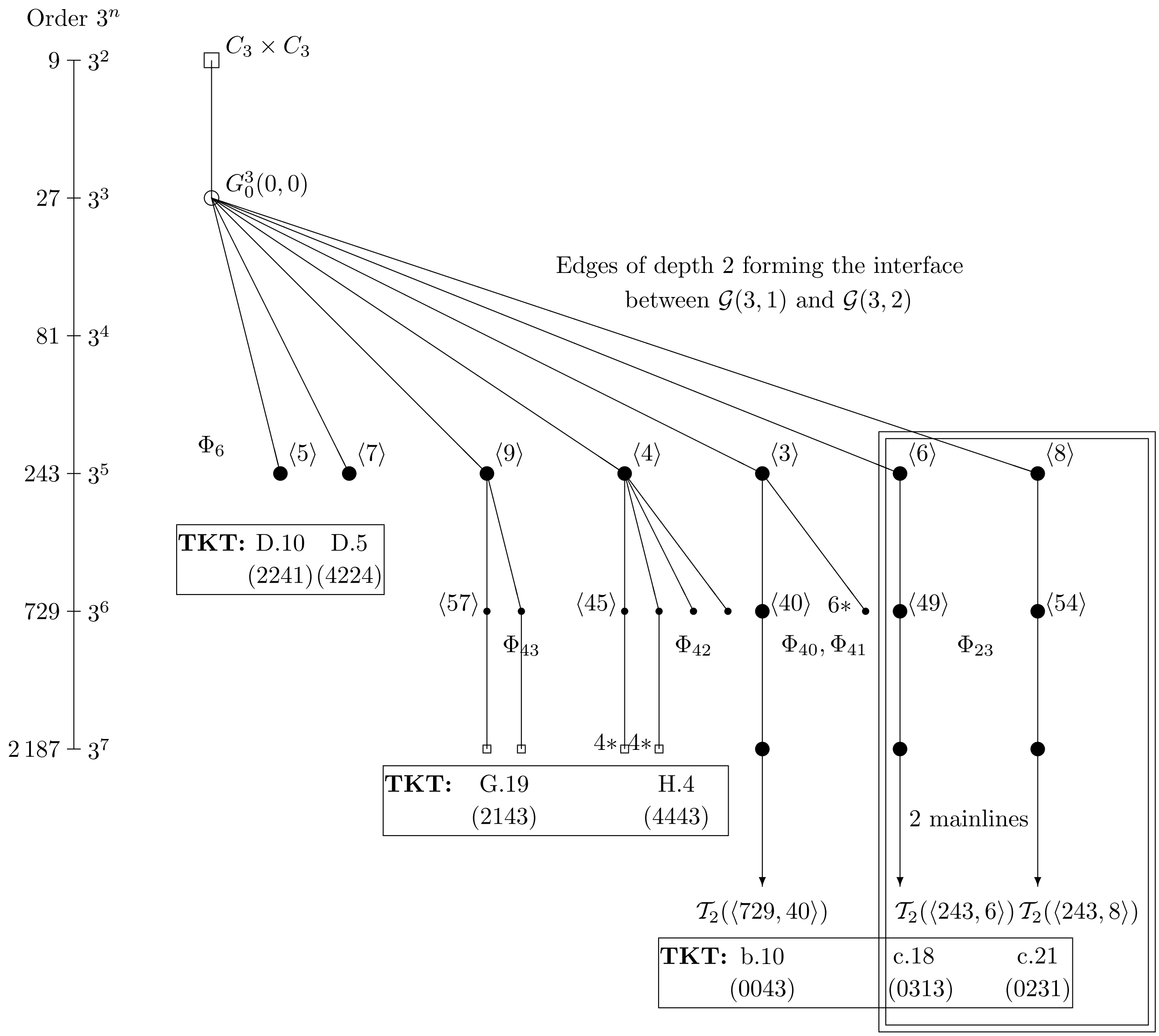
§ 4.2. TKT-Pruning $\mathcal{G}(3,2)$

The top vertices $\langle 243,5\rangle$ and $\langle 243,7\rangle$ are terminal metabelian Schur σ-groups without descendants. Descendants of $\langle 243,9\rangle$, resp. $\langle 243,4\rangle$, share a fixed TKT G.19, resp H.4. And the TKT of all descendants of $\langle 243,3\rangle$ must contain a transposition, which is not the case for TKTs in sections c and E. Therefore, only descendants of $\langle 243,6\rangle$ and $\langle 243,8\rangle$ can have TKTs in sections c and E.

FIGURE 5. TKT-pruning the top of coclass graph $\mathcal{G}(3,2)$



The top vertices $\langle 243, 5\rangle$ and $\langle 243, 7\rangle$ are terminal metabelian Schur σ-groups without descendants. Descendants of $\langle 243, 9\rangle$, resp. $\langle 243, 4\rangle$, share a fixed TKT G.19, resp H.4. And the TKT of all descendants of $\langle 243, 3\rangle$ must contain a transposition, which is not the case for TKTs in sections c and E. Therefore, only descendants of $\langle 243, 6\rangle$ and $\langle 243, 8\rangle$ can have TKTs in sections c and E.

§ 4.3. TKT-Pruning $\mathcal{T}_2(\langle 243, 8\rangle)$

Definition 4.1.

The *TKT-pruned descendant tree* $\mathcal{T}^*(\langle 243, 8\rangle)$ consists of all descendants G of the root $\langle 243, 8\rangle$ such that

(1) $\varkappa(G)$ is one of the TKTs c.21 or E.8 or E.9 (that is, we cancel all the trash with TKT G.16),

(2) if $\varkappa(G)$ is of TKT c.21 then G has descendants, (i.e., we omit terminal vertices with TKT c.21),

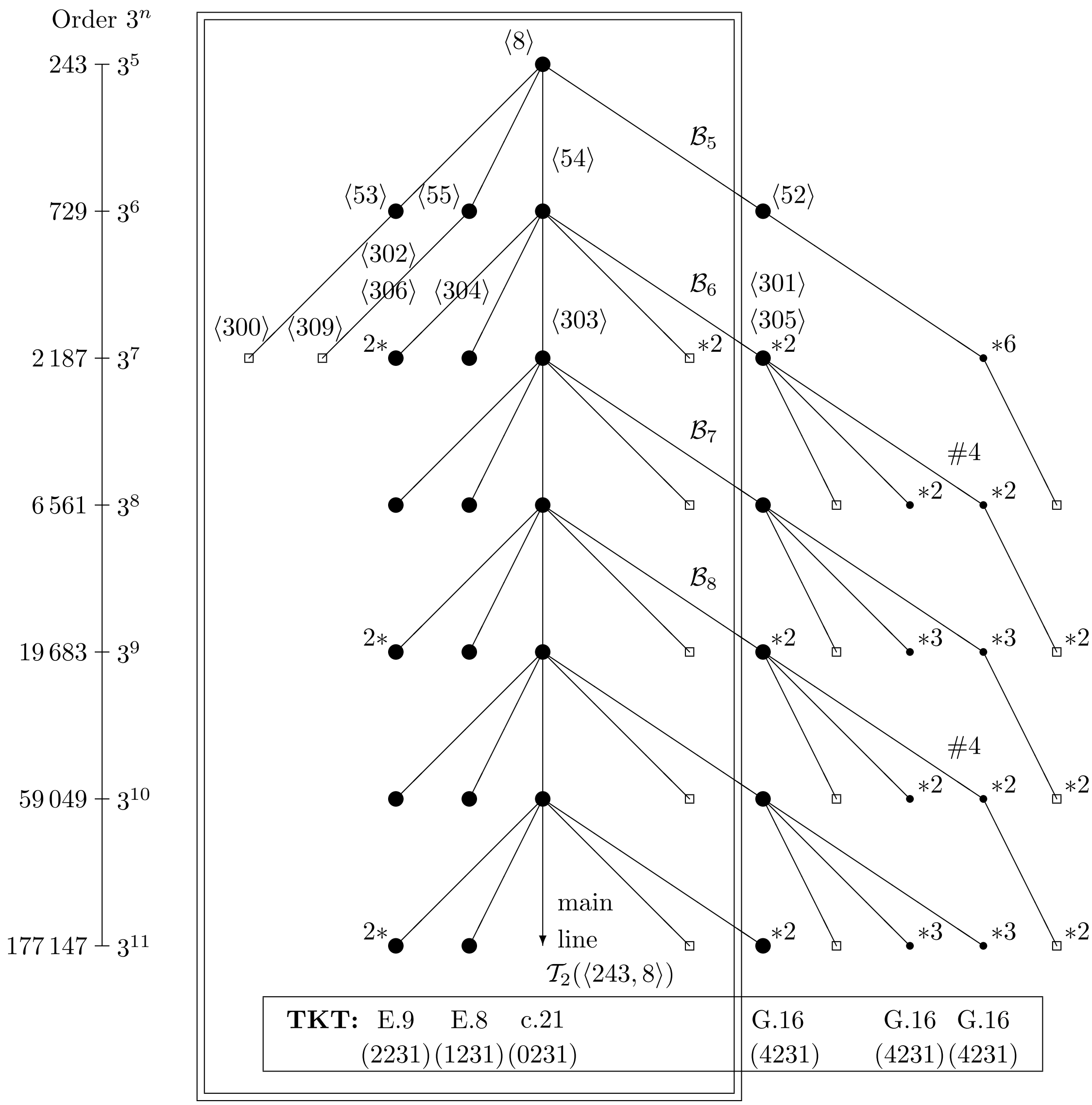
(3) G is a σ-group.

(See Figures 7,8.)

Remark 4.1.

The motivation for defining $\mathcal{T}^*(\langle 243, 8\rangle)$ is that Brink and Gold indicated a possible length $\ell_3(K) \geq 3$ for the field $K = \mathbb{Q}(\sqrt{-9748})$ with TKT E.9 for which Scholz and Taussky had claimed $\ell_3(K) = 2$.
(See [2], [3], and page 41 in [8].)

FIGURE 6. TKT-pruning the coclass tree $\mathcal{T}_2(\langle 243, 8\rangle)$

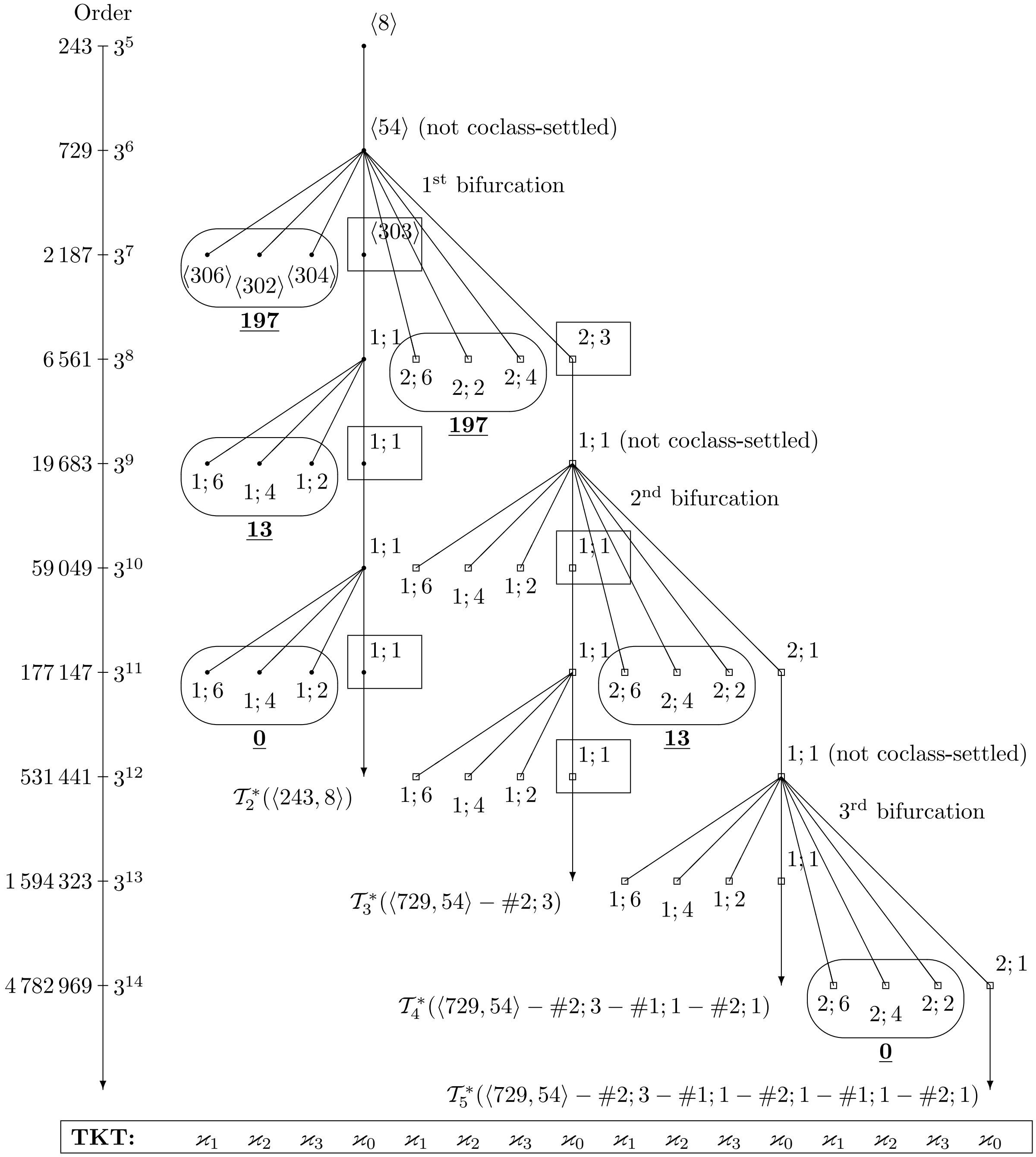


§ 4.4. Construction of $\mathcal{T}^*(\langle 243, 8\rangle)$

Here we also prune the tree from vertices with TKT c.21 at depth 1 with respect to the mainlines, which are terminal and do not give rise to further descendants. The TKTs are briefly denoted by
$\varkappa_1 = (2334) \sim \varkappa_2 = (2434)$ E.9,
$\varkappa_3 = (2234)$ E.8,
$\varkappa_0 = (2034)$ c.21.

The bifurcation at $\langle 729, 54\rangle$ has not been investigated further in previous papers, since Ascione restricted her trees to coclass 2 and Nebelung devoted her attention to metabelian 3-groups.

FIGURE 7. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 8\rangle)$ restricted to σ-groups with balanced covers in ovals, Brink/Gold's groups in rectangles, and identifiers of SmallGroups and ANUPQ



Here we also prune the tree from vertices with TKT c.21 at depth 1 with respect to the mainlines, which are terminal and do not give rise to further descendants. The TKTs are briefly denoted by
$\varkappa_1 = (2334) \sim \varkappa_2 = (2434)$ E.9,
$\varkappa_3 = (2234)$ E.8,
$\varkappa_0 = (2034)$ c.21.

§ 4.5. Biperiodic Structure of $\mathcal{T}^*(\langle 243, 8\rangle)$

We consider the intersections of $\mathcal{T}^*(\langle 243, 8\rangle)$ with coclass graphs $\mathcal{G}(3, r)$. We put

$$\mathcal{T}_2^*(\langle 243, 8\rangle) = \mathcal{T}^*(\langle 243, 8\rangle) \cap \mathcal{G}(3, 2)$$

and, for all $r \geq 3$,

$$\mathcal{G}_r^*(\langle 243, 8\rangle) = \mathcal{T}^*(\langle 243, 8\rangle) \cap \mathcal{G}(3, r).$$

Theorem 4.1. (*First periodicity*).
(See Figures 6 and 7,8.)

(1) $\mathcal{T}_2^*(\langle 243, 8\rangle)$ is a subtree of $\mathcal{T}^*(\langle 243, 8\rangle)$.
(2) All vertices are metabelian and unbalanced.
(3) Vertices of TKT c.21 form an infinite mainline with unique group $M_n^{(2)}$ of each order 3^n, $n \geq 5$.
(4) Every branch is of depth 1 and contains two groups $G_{n,1}^{(2)}, G_{n,2}^{(2)}$ of TKT E.9 and a single group $G_{n,3}^{(2)}$ of TKT E.8, each of order 3^n with odd $n \geq 7$.

Theorem 4.2. (*Second periodicity*).
(See Figures 7,8.)
For $3 \le r \le 5$,

(1) the graph $\mathcal{G}_r^*(\langle 243, 8\rangle)$ consists of
3 isolated vertices $S_k^{(r)}$, $1 \le k \le 3$,
and a subtree $\mathcal{T}_r^*(M_{3r-1}^{(r)})$ of $\mathcal{T}^*(\langle 243, 8\rangle)$,

(2) $\mathcal{T}_r^*(M_{3r-1}^{(r)})$ is isomorphic to $\mathcal{T}_2^*(\langle 243, 8\rangle)$ as a graph, and additionally, the two trees share the same distribution of TKTs,

(3) all vertices G of $\mathcal{G}_r^*(\langle 243, 8\rangle)$ are
non-metabelian of derived length $\mathrm{dl}(G) = 3$ with
cyclic second derived subgroup G'' of order 3^{r-2}
contained in the centre $\zeta_1(G)$ of type $(3, 3^{r-1})$,

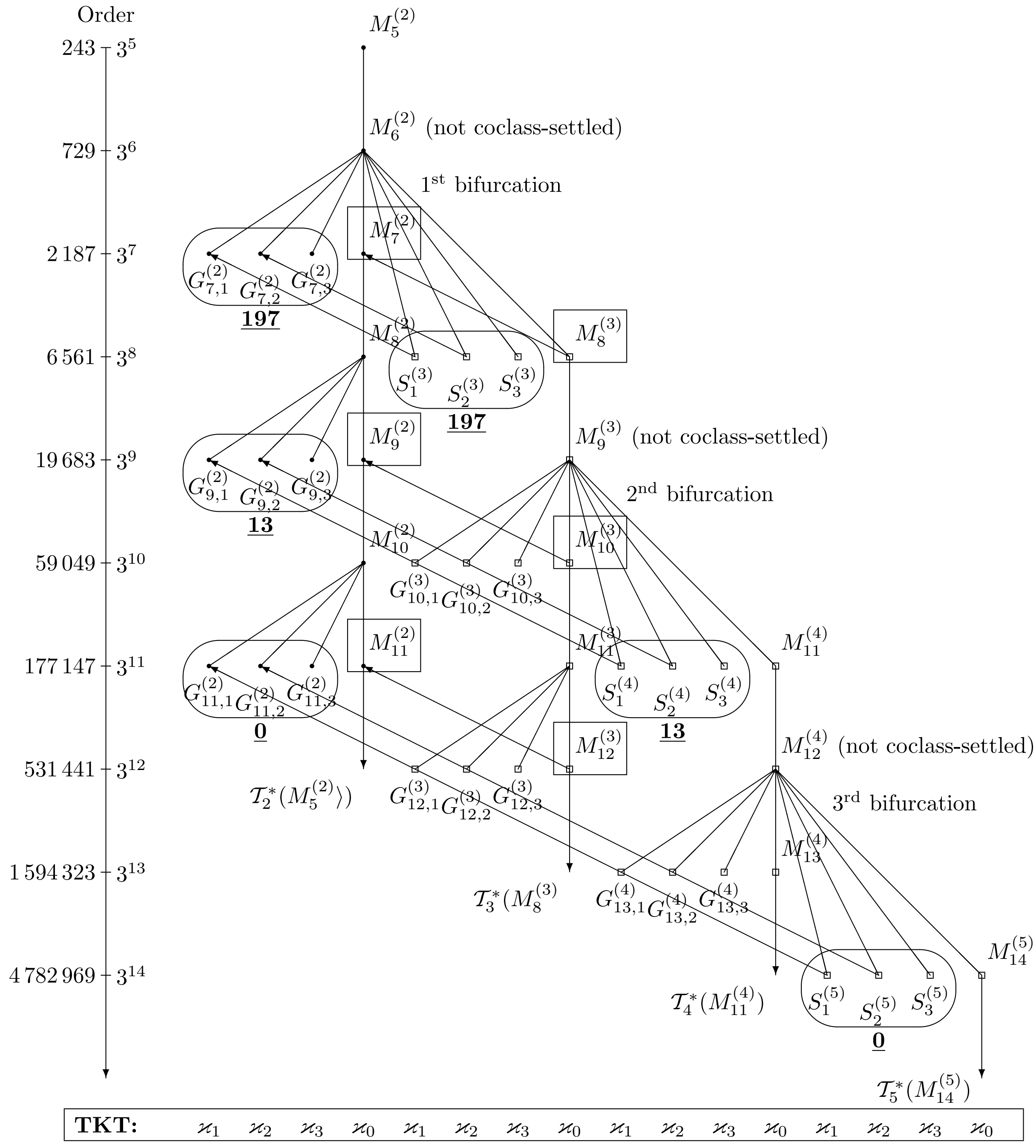
(4) the tree root $M_{3r-1}^{(r)}$ and the isolated vertices $S_k^{(r)}$ are of order 3^{3r-1},

(5) only the isolated vertices $S_k^{(r)}$ are Schur σ-groups,
two of them $S_1^{(r)}, S_2^{(r)}$ have TKT E.9,
and the remaining one $S_3^{(r)}$ has TKT E.8,

(6) each $S_k^{(r)}$ is the unique element in the balanced cover $\mathrm{cov}_*(G_{2r+1,k}^{(2)})$ of the branch group $G_{2r+1,k}^{(2)}$ of order 3^{2r+1} on the tree $\mathcal{T}_2^*(\langle 243, 8\rangle)$.

Conjecture 4.2.
Theorem 4.2 is also correct for any $r \ge 6$.

FIGURE 8. TKT-pruned descendant tree $\mathcal{T}^*(\langle 243, 8\rangle)$ restricted to σ-groups with balanced covers in ovals, Brink/Gold's groups in rectangles, projections to the metabelianizations, and formal identifiers



The techniques for reaching the targets of this presentation are based on the results of

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