On sets of integers which contain no three terms in geometric progression

Nathan McNew
Dartmouth College
Hanover, New Hampshire

INTEGERS Conference
October 27, 2013

Arithmetic Progressions

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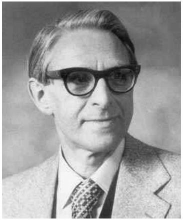
$$\text{Upper Density} \qquad \bar{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1,N]|}{N}$$

$$\text{Density} \qquad d(A) = \lim_{N\to\infty} \frac{|A \cap [1,N]|}{N}$$

Roth's Theorem



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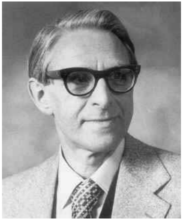

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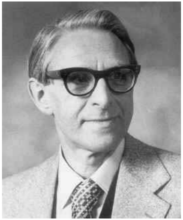

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How large of a set can we construct while avoiding 3-term APs?

Sets free of 3-term-arithmetic progressions

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- $\frac{N(\log\log N)^{5}}{\log N}$ (Sanders, 2010)

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Using this, Rankin constructs the set

$$G_3^* = \{n \in \mathbb{N} : \text{for all primes } p, v_p(n) \in A_3^*\}$$

which is free of geometric progressions. (A_3^* is the set free of arithmetic progressions obtained by the greedy algorithm.)

Rankin's Set

$$\begin{aligned} G_3^* &= \{n \in \mathbb{N} : \text{for all primes } p, v_p(n) \in A_3^*\} \\ &= \{1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19 \cdots\} \end{aligned}$$

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Rankin's set is the set obtained by greedily including integers without creating a geometric progression, and its density is

$$d(G_3^*) = \prod_p \left(\frac{p-1}{p} \sum_{i \in A_3^*} \frac{1}{p^i} \right) = \frac{1}{\zeta(2)} \prod_{i>0} \frac{\zeta(3^i)}{\zeta(2 \cdot 3^i)} = 0.71974\ldots$$

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What is the greatest possible density of a geometric progression free set?

Notation

Define:

$$\overline{\alpha} = \sup\{\bar{d}(A) : A \subset \mathbb{N} \text{ is GP-free}\}$$
$$\alpha = \sup\{d(A) : A \subset \mathbb{N} \text{ is GP-free and } d(A) \text{ exists}\}$$

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Proof.

For any N, let $k \leq N/4$ be odd. A GP-free set cannot contain all of $k, 2k$ and $4k$. These triples do not overlap so this excludes $N/8$ numbers less than N from the set. □

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Theorem (M., 2013)

The constant $\overline{\alpha}$ is effectively computable, and satisfies

$$0.730027 < \overline{\alpha} < 0.772059.$$

Avoiding s-smooth progressions

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Theorem (Nathanson and O'Bryant, 2013)

$\overline{\alpha_2} = 0.846378\ldots$ *(and is irrational.)*

Bounding $\overline{\alpha_3}$

Idea: the first seven 3-smooth numbers, $\{1, 2, 3, 4, 6, 8, 9\}$, contain the 4 progressions $(1, 2, 4), (2, 4, 8), (1, 3, 9)$ and $(4, 6, 9)$ which cannot all be precluded by removing any single number.

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Thus each $b \leq N/9$ with $(b, 6) = 1$ requires 2 exclusions, removing $2N/27$ potential numbers from a 3-smooth GP-free set.

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Thus each $b \leq N/9$ with $(b, 6) = 1$ requires 2 exclusions, removing $2N/27$ potential numbers from a 3-smooth GP-free set.

In general: Compute the largest subset of the 3-smooth integers up to k free of geometric progressions. If it requires an additional number to be excluded to avoid 3-smooth GPs, we get a better upper bound for $\overline{\alpha_3}$.

Bounding $\overline{\alpha_3}$

k	# of exclusions		k	# of exclusions		k	# of exclusions
4	1		243	13		1458	25
9	2		256	14		1728	26
16	3		288	15		1944	27
18	4		384	16		2048	28
32	5		486	17		2304	29
36	6		512	18		2592	30
64	7		576	19		3072	31
81	8		729	20		3888	32
96	9		864	21		4096	33
128	10		972	22		4374	34
144	11		1024	23		5184	35
192	12		1296	24		5832	36

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$$\overline{\alpha_3} < 1 - \frac{1}{3}\left(\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{18}+\frac{1}{32}+\cdots+\frac{1}{5832}\right) \approx 0.791266$$

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$$0.734133 < \overline{\alpha_7} < 0.772059$$

Lower Bounds

We can use lower bounds for $\overline{\alpha_s}$ to create GP-free sets with greater upper density than Rankin's set.

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Idea: Use the $\overline{\alpha_s}$ construction for primes at most s, and stitch this together with Rankin's construction for primes greater than s.

Theorem

$$\overline{\alpha_s} \prod_{p>s} \left(\frac{p-1}{p} \sum_{i \in A_3^*} p^{-i} \right) \leq \overline{\alpha} \leq \overline{\alpha_s}$$

So, $\lim_{s \to \infty} \overline{\alpha_s} = \overline{\alpha}$.

Computing $\overline{\alpha}$

Theorem (M., 2013)

For each ϵ with $0 < \epsilon < 1$, the constant $\overline{\alpha}$ can be computed to within ϵ in time

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The same is true for

$$\left(\frac{N}{48}, \frac{N}{45}\right] \bigcup \left(\frac{N}{40}, \frac{N}{36}\right] \bigcup \left(\frac{N}{32}, \frac{N}{27}\right] \bigcup \left(\frac{N}{24}, \frac{N}{12}\right] \bigcup \left(\frac{N}{9}, \frac{N}{8}\right] \bigcup \left(\frac{N}{4}, N\right]$$

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$$\left(\frac{N}{48}, \frac{N}{45}\right] \bigcup \left(\frac{N}{40}, \frac{N}{36}\right] \bigcup \left(\frac{N}{32}, \frac{N}{27}\right] \bigcup \left(\frac{N}{24}, \frac{N}{12}\right] \bigcup \left(\frac{N}{9}, \frac{N}{8}\right] \bigcup \left(\frac{N}{4}, N\right]$$

Theorem (M., 2013)

$$0.815509 < \overline{\beta} < 0.819222$$

Integer Ratios

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Theorem (M., 2013)

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Theorem (M., 2013)

$$\beta > 0.72195 > d(G_3^*) \approx 0.71974$$

Open Questions

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- Must every infinite set of natural numbers with bounded gaps between consecutive terms contain arbitrarily long geometric progressions?

The end

Thank you