Weierstrass points on Drinfeld modular curves

Christelle Vincent

Stanford University

Organization of the talk

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2. The Drinfeld setting

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3. The modular Wronskian

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4. On-going work

Weierstrass points; a motivating example

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- No function with a single pole at a point and regular elsewhere.

- For most points Q of X, also no function with a double pole at Q and regular elsewhere.

  For every branch point P of f, there is a function F that has a double pole at P and is regular elsewhere.

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More generally, let X be a curve of genus $g \geq 2$.

For P on X, is there a nonzero rational function with a pole of order exactly n at P and regular elsewhere?

- If the answer is yes, we say that n is a *pole number* at P.
- If the answer is no, we say that n is a *gap number* at P.

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We define the Weierstrass weight to be:

$$\mathrm{wt}(P) = \sum_{i=1}^{g} n_i(P) - n_i.$$

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If k is of positive characteristic, the gap sequence could be different.

Why is this important?

If X is a curve over a finite field, $g \geq \sqrt{q}$, and X has a classical gap sequence, then the upper bound of RH can be improved.

For example, a curve of genus 3 over $\mathbb{F}_9$ with classical gap sequence can have at most 24 rational points, but

$$y^3 + y + x^4 = 0$$

has 28 rational points.

Using Riemann-Roch

Let C be a canonical divisor on X. By Riemann-Roch, we have

$$\dim(L(nP)/L((n-1)P)) = 1 - \dim(L(C-(n-1)P)/L(C-nP)).$$

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In other words, n is a gap at P if and only if

$$L(C-(n-1)P) \supsetneq L(C-nP)$$

i.e. there is a 1-form ω such that $v_P(\omega) = n-1$.

Notation

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$K = \mathbb{F}_q(T)$	$(\mathbb{Q})$
$A = \mathbb{F}_q[T]$	$(\mathbb{Z})$
$K_\infty = \mathbb{F}_q((T^{-1}))$	$(\mathbb{R})$
$C = \hat{\bar{K}}_\infty$	$(\mathbb{C})$
$\Omega = C - K_\infty$	$(\mathbb{H})$
$\mathrm{GL}_2(A)$	$(\mathrm{SL}_2(\mathbb{Z}))$
$\mathfrak{p} = \langle \pi \rangle$	(ℓ)
π prime of degree d	

Drinfeld modules

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A Drinfeld module of rank 2 over L is a ring homomorphism $\phi\colon A \to \mathrm{End}_L(\mathbb{G}_a)$ such that

$$\phi(T) = T\tau^0 + g\tau + \Delta\tau^2$$

where $\tau\colon x \mapsto x^q$ and $\Delta \neq 0$.

Important facts

The Drinfeld modules of rank 2 will play the role of elliptic curves:

- Over $A/\mathfrak{p}$, there is a notion of *supersingular* Drinfeld module.
- $\phi[n](L) = \{x \in L : \phi(n)(x) = 0\}$

  $X_0(\mathfrak{n})$ is a coarse moduli space for Drinfeld modules with level-n structure.

The modular curve $X_0(\mathfrak{n})$

As in the classical case,

- We define the congruence subgroup

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \,|\, c \equiv 0 \pmod{\mathfrak{n}} \right\},$$

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$$X_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n}) \backslash \Omega \cup \{\text{finitely many cusps}\}.$$

- $X_0(\mathfrak{n})$ is a complete, smooth, irreducible curve defined over K.

Drinfeld modular forms

Definition

A function $f : \Omega \to C$ is called a *Drinfeld modular form of weight k and type l for Γ*, where $k \geq 0$ is an integer and l is a class in $\mathbb{Z}/(\#\det\Gamma)$, if

1. for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $f(\gamma z) = (\det\gamma)^{-l}(cz+d)^k f(z)$;
2. f is rigid analytic;
3. f is "holomorphic at the cusps".

Upshot

Double cuspidal Drinfeld modular forms of weight 2 on $\Gamma_0(\mathfrak{n})$ correspond to holomorphic differentials on $X_0(\mathfrak{n})$.

Weierstrass points on $X_0(\mathfrak{p})$

Theorem (Baker)

Let $P = (\phi, H)$ be a Weierstrass point on $X_0(\mathfrak{p})$. Then the reduction of ϕ modulo $\mathfrak{p}$ is supersingular.

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Can we say anything more?

A theorem in the classical setting

Building on work of Rohrlich's,

Theorem (Ahlgren-Ono)

Let $\ell \geq 23$ and g_ℓ be the genus of $X_0(\ell)$. Then

$$\prod_{Q \in X_0(\ell)} (x - j(Q))^{\mathrm{wt}(Q)} \equiv \prod_{\substack{E/\overline{\mathbb{F}}_\ell \\ E \text{ supersingular}}} (x - j(E))^{g_\ell(g_\ell - 1)} \pmod{\ell}.$$

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Furthermore, if C is a canonical divisor on $X_0(\mathfrak{p})$, then $L(C)$ corresponds to the space $M^2_{2,1}(\Gamma_0(\mathfrak{p}))$.

This correspondence respects orders of vanishing in a way that can be made precise.

The modular Wronskian

We define the following modular form on $\Gamma_0(\mathfrak{p})$:

Definition

Let $\{f_1, f_2, ...f_{g_\mathfrak{p}}\}$ be a basis for $M^2_{2,1}(\Gamma_0(\mathfrak{p}))$. Consider the Wronskian

$$W(f_1, \ldots, f_{g_\mathfrak{p}}) = \begin{vmatrix} f_1(z) & \ldots & D_{g_\mathfrak{p}-1}(f_1)(z) \\ \vdots & & \vdots \\ f_{g_\mathfrak{p}}(z) & \ldots & D_{g_\mathfrak{p}-1}(f_{g_\mathfrak{p}})(z) \end{vmatrix}.$$

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This is a modular form of weight $g_\mathfrak{p}(g_\mathfrak{p}+1)$ for $\Gamma_0(\mathfrak{p})$. We denote by $W(z)$ the normalization of this form that has 1 as its leading coefficient for the u-series expansion at ∞.

Weierstrass points and the modular Wronskian

Proposition

For any point P of $X_0(\mathfrak{p})$, we have

$$\mathrm{ord}_P(W(z)(dz)^{g_{\mathfrak{p}}(g_{\mathfrak{p}}+1)/2}) \geq \mathrm{wt}(P).$$

In addition, when P is not elliptic and is not a Weierstrass point, or P is not a cusp, we have equality: $\mathrm{ord}_P(W(z)(dz)^{g_{\mathfrak{p}}(g_{\mathfrak{p}}+1)/2}) = 0$.

The Drinfeld setting

Weierstrass points

The modular Wronskian

4. On-going work

The main idea

We want to study the divisor of $W(z)$, or at least its divisor modulo $\mathfrak{p}$. This is hard because $W(z)$ is modular for $\Gamma_0(\mathfrak{p})$.

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However, studying divisors of forms for $\mathrm{GL}_2(A)$ is much easier.

Getting down to $\mathrm{GL}_2(A)$

Two ways:

$$\mathrm{Tr}(f) = \sum_{\gamma \in \Gamma_0(\mathfrak{p}) \backslash \mathrm{GL}_2(A)} f(z) \| [\gamma].$$

$$\mathrm{N}(f) = \prod_{\gamma \in \Gamma_0(\mathfrak{p}) \backslash \mathrm{GL}_2(A)} f(z) \| [\gamma].$$

Congruence for the trace map

Let

$$\widetilde{\mathrm{Tr}(f)} = \mathrm{Tr}(fg_{(0)}),$$

where

$$g_{(0)} = g_d - \pi^{(q^d-1)/2} g_d|[W_\mathfrak{p}] \equiv 1 \pmod{\mathfrak{p}}.$$

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But... this destroys the divisor.

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Theorem (V.)

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This preserves the divisor.

Consequence

Under the assumption that $W(z)$ is an eigenform for the Atkin-Lehner involution, we have

$$\widetilde{\mathrm{N}(W)} \equiv (\widetilde{\mathrm{Tr}(W)})^2 \pmod{\mathfrak{p}}.$$

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The form on the LHS has weight $(q^d + 1)g_{\mathfrak{p}}(g_{\mathfrak{p}} + 1)$.

The form on the RHS has weight $2g_{\mathfrak{p}}(q^d + g_{\mathfrak{p}})$.

Forms with higher weight than filtration

By Dobi-Wage-Wang,we have:

If P is a supersingular point then

$$v_P(\widetilde{\mathrm{N}(W)}) \geq g_{\mathfrak{p}}(g_{\mathfrak{p}} - 1).$$

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What's more, we should be able to know the Weierstrass points modulo $\mathfrak{p}$ up to the divisor of this mysterious $\widetilde{\mathrm{Tr}(W)}$.

A partial result

In very specific cases ($q = p > 2, d = 3$), we can obtain an analogue of a theorem of Rohrlich's, which amounts to saying

$$\widetilde{\mathrm{Tr}(W)} = (-1)^{(q+1)/2} g^{\frac{q^2(q-1)}{2}} h^{\frac{q^2(q+1)}{2}}.$$

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In general, we expect that this $\widetilde{\mathrm{Tr}(W)}$ should be hard to compute explicitly.

Thank you!