On the ratios of consecutive divisors

Andreas Weingartner

Department of Mathematics
Southern Utah University

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The ratios of consecutive divisors

**Example:** The divisors of \( n = 2013 = 3 \cdot 11 \cdot 61 \) are

\[
\{d_1, d_2, \ldots, d_8\} = \{1, 3, 11, 33, 61, 183, 671, 2013\}.
\]
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\[ \left\{ \frac{d_{i+1}}{d_i} : 1 \leq i < 8 \right\} = \left\{ \frac{3}{1}, \frac{11}{3}, \frac{33}{11}, \frac{61}{33}, \frac{183}{61}, \frac{671}{183}, \frac{2013}{671} \right\} = \left\{ 3, \frac{11}{3}, \frac{61}{33}, 3, \frac{11}{3}, 3 \right\} . \]
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\]

\[
= \left\{ 3, \frac{11}{3}, 3, \frac{61}{33}, 3, \frac{11}{3}, 3 \right\}
\]

The maximum and minimum ratios are

\[
R(2013) := \max_{1 \leq i < 8} \frac{d_{i+1}}{d_i} = \frac{11}{3},
\]

\[
r(2013) := \min_{1 \leq i < 8} \frac{d_{i+1}}{d_i} = \frac{61}{33}.
\]
Maximum and minimum ratios for small $n$

$$R(n) := \max_{1 \leq i < \tau(n)} \frac{d_{i+1}}{d_i}, \quad r(n) := \min_{1 \leq i < \tau(n)} \frac{d_{i+1}}{d_i}.$$
The minimum ratio of consecutive divisors

Let

\[ r(n) = \min_{1 \leq i < \tau(n)} \frac{d_{i+1}}{d_i}, \]

\[ S(x, t) := |\{ n \leq x : r(n) \geq t \}|. \]
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Erdős conjectured in the 1940s that for all fixed \( t > 1 \),

\[ S(x, t) = o(x) \]

This was proved in 1984 by Maier and Tenenbaum.
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A result for the case \( t = 2 \) by Stef is

\[ \frac{x}{(\log x)^{\beta + o(1)}} \leq S(x, 2) \leq x e^{-c\sqrt{\log \log x}} \]

where \( \beta = 0.00415\ldots \), \( c > 0 \) is some constant.
An alternate formula for the maximum ratio

For \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \), \( p_1 < \ldots < p_k \), define

\[
F(n) = \max_{1 \leq i \leq k} p_i (p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k}).
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Tenenbaum (1986) showed that

\[
\frac{F(n)}{n} = \max_{1 \leq i < \tau(n)} \frac{d_{i+1}}{d_i} \quad (n \geq 2).
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**Example:** For \( n = 2013 = 3 \cdot 11 \cdot 61 \) we have

\[
F(n) = \max\{3^2 \cdot 11 \cdot 61, \ 11^2 \cdot 61, \ 61^2\} = 11^2 \cdot 61
\]

and

\[
\frac{F(2013)}{2013} = \frac{11^2 \cdot 61}{3 \cdot 11 \cdot 61} = \frac{11}{3}.
\]
The number of $n \leq x$ with $t$-dense divisors.

Define

$$D(x, t) := \# \left\{ n \leq x : \max_{1 \leq i < \tau(n)} \frac{d_{i+1}}{d_i} \leq t \right\} = \# \left\{ n \leq x : \frac{F(n)}{n} \leq t \right\} = "The number of n \leq x with t-dense divisors"$$
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= "The number of $n \leq x$ with $t$-dense divisors"

Tenenbaum (1986): For $t \geq e^{(\log \log x)^{5/3} + \epsilon}$

$$x \frac{\log t}{\log x} \ll D(x, t) \ll x \frac{\log t \log \left( \frac{2 \log x}{\log t} \right)}{\log x}.$$
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Saias (1997): For $x \geq t \geq 2$,

$$D(x, t) \ll \| x \frac{\log t}{\log x} \|.$$
Application: Longest path in the divisor graph

The divisor graph of order $n$ is the graph with vertices \{1, 2, \ldots, n\}, and an edge between $a$ and $b$ iff $a|b$ or $b|a$.

Let $f(n) :=$ length of the longest path in the divisor graph of order $n$.

Example: $f(16) = 14$:

10 → 5 → 15 → 3 → 6 → 12 → 4 → 8 → 16 → 2 → 14 → 7 → 1 → 9.
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**Example:** $f(16) = 14$:

$10 \rightarrow 5 \rightarrow 15 \rightarrow 3 \rightarrow 6 \rightarrow 12 \rightarrow 4 \rightarrow 8 \rightarrow 16 \rightarrow 2 \rightarrow 14 \rightarrow 7 \rightarrow 1 \rightarrow 9$. 
Application: Longest path in the divisor graph

$f(n)$ is the length of the longest path in the divisor graph of order $n$.

Pollington (1983):

$$f(n) \geq n \exp \left\{ - (2 + o(1)) \sqrt{\log n \log \log n} \right\}$$

Tenenbaum (1995):

$$\tilde{D}(n/4, 2) \leq f(n) \leq 2D(n, (\log n)^5)$$

Saias (1998):

$$f(n) \ll \frac{n}{\log n}$$
An asymptotic formula for $D(x, t)$

Theorem (2003)

Uniformly for $x \geq t \geq \exp \left\{ (\log \log x)^{5/3 + \varepsilon} \right\}$,

$$D(x, t) = x d(v) \left\{ 1 + O \left( \frac{1}{\log t} \right) \right\},$$

where

$$v = \frac{\log x}{\log t}$$

and $d(v)$ is a continuous, decreasing function which satisfies

$$1.8 \leq (v + 1) d(v) \leq 3.4 \quad (v \geq 1).$$
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**Question**: Does $\lim_{v \to \infty} (v + 1) \, d(v)$ exist?
\[ D(x, t) \sim x \, d \left( \frac{\log x}{\log t} \right) \]

**Figure:** A graph of \( d(\nu) \).
An asymptotic formula for $d(v)$.

**Theorem (2013)**

For $v \geq 1$ we have

$$d(v) = \frac{C}{v + 1} \left\{ 1 + O(v^{-2}) \right\},$$

where $C = \frac{1}{1 - e^{-\gamma}} = 2.280291\ldots$, and $\gamma = 0.577215\ldots$ is Euler’s constant.

Remark: The error term is almost best possible: the theorem would be false if $O(v^{-2})$ was replaced by $O(v^{-2.1})$. 
An asymptotic formula for $d(v)$.

**Theorem (2013)**

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**Remark:** The error term is almost best possible: the theorem would be false if $O(v^{-2})$ was replaced by $O(v^{-2.1})$. 
\[(v + 1) d(v) \sim \frac{1}{1 - e^{-\gamma}}\]

**Figure:** A graph of \((v + 1)d(v)\) and its limit \(C = 1/(1 - e^{-\gamma})\).
Combining $D(x, t) \sim x d(v)$ with $d(v) \sim \frac{C}{v+1}$

**Corollary**

Uniformly for $x \geq 3$, $x \geq t \geq \exp\left\{ (\log \log x)^{5/3+\varepsilon}\right\}$, we have

$$D(x, t) = \frac{Cx \log t}{\log xt} \left\{ 1 + O \left( \frac{1}{\log t} + \frac{\log^2 t}{\log^2 x} \right) \right\}.$$
Combining $D(x, t) \sim x d(v)$ with $d(v) \sim \frac{C}{v+1}$

Corollary

*Uniformly for $x \geq 3$, $x \geq t \geq \exp \left\{ \left( \log \log x \right)^{5/3+\varepsilon} \right\}$, we have*

$$D(x, t) = \frac{C x \log t}{\log xt} \left\{ 1 + O \left( \frac{1}{\log t} + \frac{\log^2 t}{\log^2 x} \right) \right\}.$$

Under the Riemann hypothesis, this holds for $t \geq (\log x)^{4+\varepsilon}$. However,

$$D(x, t) \sim \frac{C x \log t}{\log xt} \quad (x \to \infty)$$

*can not* hold in general for fixed $t$ since

$$D(x, p) - D(x, p-0) \ll_{p} \frac{x}{\log x},$$

where the last estimate is due to Saias.
Proof of $d(n) \sim \frac{C}{v+1}$: A new functional equation

Define

$$\chi_t(n) = \begin{cases} 
1 & \text{if } n \text{ has } t\text{-dense divisors,} \\
0 & \text{else.}
\end{cases}$$
Proof of $d(v) \sim \frac{c}{v+1}$: A new functional equation

Define

$$\chi_t(n) = \begin{cases} 1 & \text{if } n \text{ has } t\text{-dense divisors,} \\ 0 & \text{else.} \end{cases}$$

Let

$$\Phi(x, y) = |\{n \leq x : p|n \Rightarrow p > y\}|.$$
Proof of \( d(v) \sim \frac{C}{v+1} \): A new functional equation

Define

\[ \chi_t(n) = \begin{cases} 1 & \text{if } n \text{ has } t\text{-dense divisors,} \\ 0 & \text{else.} \end{cases} \]

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\[ \Phi(x, y) = |\{n \leq x : p|n \Rightarrow p > y\}|. \]

**Lemma**

*For* \( x \geq 1, t \geq 2 \) *we have*

\[ [x] = \sum_{1 \leq n \leq x} \chi_t(n) \Phi(x/n, nt). \]
Proof of $d(v) \sim \frac{C}{v+1}$: Consequences of new equation

We get

$$D(x, t) = O(\sqrt{x}) + \lfloor x \rfloor - \sum_{n \leq \sqrt{x/t}} \chi_t(n) \Phi(x/n, nt).$$
Proof of $d(v) \sim \frac{C}{v+1}$: Consequences of new equation

We get

$$D(x, t) = O(\sqrt{x}) + [x] - \sum_{n \leq \sqrt{x/t}} \chi_t(n) \Phi(x/n, nt).$$

With $D(x, t) \sim x d(v)$ this yields

$$d(v) = 1 - \int_0^\infty \frac{d(u)}{u + 1} \omega\left(\frac{v - u}{u + 1}\right) \, du,$$

where $\omega(u)$ is Buchstab’s function.
Proof of $d(v) \sim \frac{c}{v+1}$: The Laplace transform

The integral equation for $d(v)$ leads to the Laplace transform of $G(y) := e^y d(e^y - 1)$,

$$
\hat{G}(s) = \int_0^\infty e^{-sy} G(y) \, dy \quad (\text{Re } s > 0),
$$

namely

$$
\hat{G}(s) = \frac{1}{(s - 1)(1 + f(s)) + e^{-\gamma}},
$$

where $f$ is the entire function given by

$$
f(s) = \int_0^\infty \left( \omega(u) - e^{-\gamma} \right) \frac{du}{(u + 1)^s}.
$$
Proof of $d(v) \sim \frac{C}{v+1}$: Inversion of the Laplace transform

Let $P_a$ denote the finite set of poles of $\hat{G}(s)$ with $-a < \Re s \leq 0$.

$$e^y d(e^y - 1) =: G(y) = \sum_{s_k \in P_a} \text{Res} \left( \hat{G}(s)e^{ys}; s_k \right) + O_a \left( e^{-ay} \right).$$

**Figure:** A contour plot of $|\hat{G}(s)|$. 
Open problems

For $t$ fixed, is

$$D(x, t) \sim C(t) \frac{x \log t}{\log x} \quad (x \to \infty)$$

for some discontinuous function $C(t)$?
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- For $t$ fixed, is
  \[ D(x, t) \sim C(t) \frac{x \log t}{\log x} \quad (x \to \infty) \]
  for some discontinuous function $C(t)$?

- Is
  \[ f(n) \sim \frac{cn}{\log n} \quad (n \to \infty) \]
  for some constant $c$, where $f(n)$ is the longest path in the divisor graph?


