On the ratios of consecutive divisors

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The ratios of consecutive divisors

Example: The divisors of $n = 2013 = 3 \cdot 11 \cdot 61$ are

$$\{d_1, d_2, \ldots, d_8\} = \{1, 3, 11, 33, 61, 183, 671, 2013\}.$$

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$$\begin{aligned} \left\{ \frac{d_{i+1}}{d_i} \,:\, 1 \le i < 8 \right\} &= \left\{ \frac{3}{1}, \frac{11}{3}, \frac{33}{11}, \frac{61}{33}, \frac{183}{61}, \frac{671}{183}, \frac{2013}{671} \right\} \\ &= \left\{ 3, \frac{11}{3}, 3, \frac{61}{33}, 3, \frac{11}{3}, 3 \right\} \end{aligned}$$

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$$= \left\{ 3, \frac{11}{3}, 3, \frac{61}{33}, 3, \frac{11}{3}, 3 \right\}$$

The maximum and minimum ratios are

$$R(2013) := \max_{1 \le i < 8} \frac{d_{i+1}}{d_i} = \frac{11}{3},$$

$$r(2013) := \min_{1 \le i < 8} \frac{d_{i+1}}{d_i} = \frac{61}{33}.$$

Maximum and minimum ratios for small n

$$R(n) := \max_{1 \le i < \tau(n)} \frac{d_{i+1}}{d_i}, \quad r(n) := \min_{1 \le i < \tau(n)} \frac{d_{i+1}}{d_i}.$$

n	$R(n)$	$r(n)$
2	2	2
4	2	2
6	2	3/2
8	2	2
9	3	3
10	5/2	2
12	2	4/3
14	7/2	2
15	3	5/3
16	2	2
18	2	3/2
20	2	5/4
21	3	7/3

The minimum ratio of consecutive divisors

Let

$$r(n) = \min_{1 \le i < \tau(n)} \frac{d_{i+1}}{d_i},$$

$$S(x,t) := |\{n \le x : r(n) \ge t\}|.$$

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Erdős conjectured in the 1940s that for all fixed $t > 1$,

$$S(x,t) = o(x)$$

This was proved in 1984 by Maier and Tenenbaum.

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A result for the case $t = 2$ by Stef is

$$\frac{x}{(\log x)^{\beta + o(1)}} \le S(x,2) \le x\, e^{-c\sqrt{\log\log x}}$$

where $\beta = 0.00415\ldots$, $c > 0$ is some constant.

An alternate formula for the maximum ratio

For $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \;\; p_1 < \ldots < p_k$, define

$$F(n) = \max_{1 \le i \le k} p_i (p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k}).$$

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Tenenbaum (1986) showed that

$$\frac{F(n)}{n} = \max_{1 \le i < \tau(n)} \frac{d_{i+1}}{d_i} \qquad (n \ge 2).$$

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Example: For $n = 2013 = 3 \cdot 11 \cdot 61$ we have

$$F(n) = \max\{3^2 \cdot 11 \cdot 61,\ 11^2 \cdot 61,\ 61^2\} = 11^2 \cdot 61$$

and

$$\frac{F(2013)}{2013} = \frac{11^2 \cdot 61}{3 \cdot 11 \cdot 61} = \frac{11}{3}.$$

The number of $n \le x$ with t-dense divisors.

Define

$$D(x,t) := \#\left\{n \le x : \max_{1\le i<\tau(n)} \frac{d_{i+1}}{d_i} \le t\right\} = \#\left\{n \le x : \frac{F(n)}{n} \le t\right\}$$

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Tenenbaum (1986): For $t \geq e^{(\log\log x)^{5/3+\epsilon}}$

$$x \frac{\log t}{\log x} \ll D(x,t) \ll x \frac{\log t \log\left(\frac{2\log x}{\log t}\right)}{\log x}.$$

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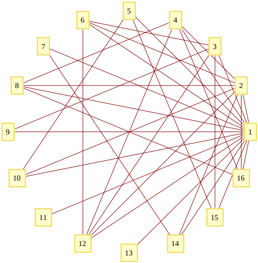
Saias (1997): For $x \geq t \geq 2$,

$$D(x,t) \ll\gg x\frac{\log t}{\log x}.$$

Application: Longest path in the divisor graph

The divisor graph of order n is the graph with vertices $\{1, 2, \ldots, n\}$, and an edge between a and b iff $a|b$ or $b|a$.



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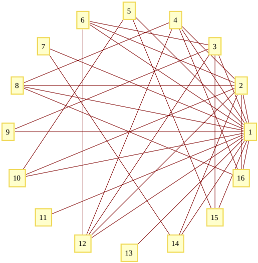

Let $f(n) :=$ length of the longest path in the divisor graph of order n.
Example: $f(16) = 14$:

$$10 \to 5 \to 15 \to 3 \to 6 \to 12 \to 4 \to 8 \to 16 \to 2 \to 14 \to 7 \to 1 \to 9.$$

Application: Longest path in the divisor graph

$f(n)$ is the length of the longest path in the divisor graph of order n.

Pollington (1983):

$$f(n) \geq n \exp\left\{-(2+o(1))\sqrt{\log n \log\log n}\right\}$$

Tenenbaum (1995):

$$\tilde{D}(n/4, 2) \leq f(n) \leq 2D(n, (\log n)^5)$$

Saias (1998):

$$f(n) \ll\gg \frac{n}{\log n}$$

An asymptotic formula for $D(x,t)$

Theorem (2003)

Uniformly for $x \ge t \ge \exp\left\{(\log\log x)^{5/3+\varepsilon}\right\}$,

$$D(x,t) = x\, d(v) \left\{ 1 + O\left(\frac{1}{\log t}\right) \right\},$$

where

$$v = \frac{\log x}{\log t}$$

and $d(v)$ *is a continuous, decreasing function which satisfies*

$$1.8 \le (v+1)\, d(v) \le 3.4 \qquad (v \ge 1).$$

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Question: Does $\lim_{v\to\infty} (v+1)\, d(v)$ exist?

$$D(x,t) \sim x\, d\left(\frac{\log x}{\log t}\right)$$

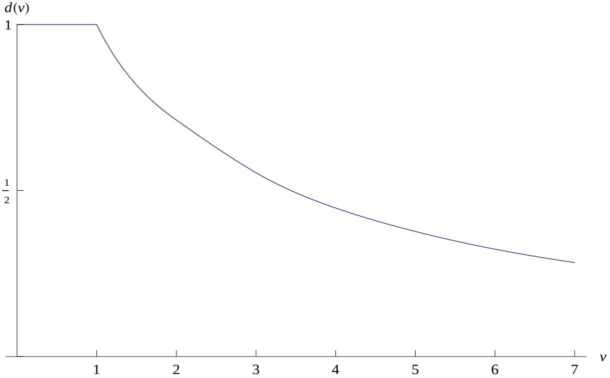

Figure: A graph of $d(v)$.

An asymptotic formula for $d(v)$.

Theorem (2013)

For $v \geq 1$ we have

$$d(v) = \frac{C}{v+1}\left\{1 + O\left(v^{-2}\right)\right\},$$

where $C = \dfrac{1}{1-e^{-\gamma}} = 2.280291...$, *and* $\gamma = 0.577215...$ *is Euler's constant.*

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Remark: The error term is almost best possible: the theorem would be false if $O(v^{-2})$ was replaced by $O(v^{-2.1})$.

$(v + 1)\, d(v) \sim \frac{1}{1-e^{-\gamma}}$

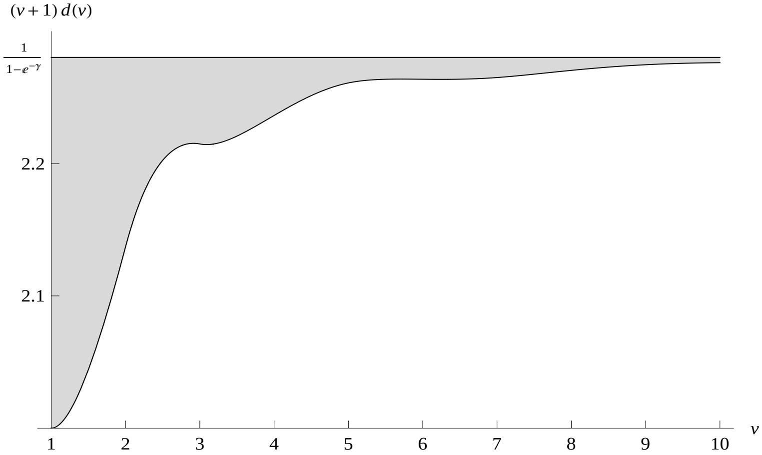

Figure: A graph of $(v+1)d(v)$ and its limit $C = 1/(1-e^{-\gamma})$.

Combining $D(x,t) \sim x\,d(v)$ with $d(v) \sim \frac{C}{v+1}$

Corollary

Uniformly for $x \geq 3$, $x \geq t \geq \exp\left\{(\log\log x)^{5/3+\varepsilon}\right\}$, *we have*

$$D(x,t) = \frac{Cx\log t}{\log xt}\left\{1 + O\left(\frac{1}{\log t} + \frac{\log^2 t}{\log^2 x}\right)\right\}.$$

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Under the Riemann hypothesis, this holds for $t \geq (\log x)^{4+\varepsilon}$.
However,

$$D(x,t) \sim \frac{Cx\log t}{\log xt} \qquad (x \to \infty)$$

can **not** hold in general for fixed t since

$$D(x,p) - D(x,p-0) \ll\gg_p \frac{x}{\log x},$$

where the last estimate is due to Saias.

Proof of $d(v) \sim \frac{C}{v+1}$: A new functional equation

Define

$$\chi_t(n) = \begin{cases} 1 & \text{if } n \text{ has } t\text{-dense divisors,} \\ 0 & \text{else.} \end{cases}$$

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Let

$$\Phi(x,y) = |\{n \le x : \; p|n \Rightarrow p > y\}|.$$

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Lemma

For $x \ge 1$, $t \ge 2$ we have

$$[x] = \sum_{1 \le n \le x} \chi_t(n)\, \Phi(x/n, nt).$$

Proof of $d(v) \sim \frac{C}{v+1}$: Consequences of new equation

We get

$$D(x,t) = O(\sqrt{x}) + [x] - \sum_{n \le \sqrt{x/t}} \chi_t(n)\, \Phi(x/n, nt).$$

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With $D(x,t) \sim x\, d(v)$ this yields

$$d(v) = 1 - \int_0^\infty \frac{d(u)}{u+1}\, \omega\left(\frac{v-u}{u+1}\right) \mathrm{d}u,$$

where $\omega(u)$ is Buchstab's function.

Proof of $d(v) \sim \frac{C}{v+1}$: The Laplace transform

The integral equation for $d(v)$ leads to the Laplace transform of $G(y) := e^y d(e^y - 1)$,

$$\widehat{G}(s) = \int_0^\infty e^{-sy} G(y)\, \mathrm{d}y \quad (\mathrm{Re}\, s > 0),$$

namely

$$\widehat{G}(s) = \frac{1}{(s-1)(1+f(s)) + e^{-\gamma}},$$

where f is the entire function given by

$$f(s) = \int_0^\infty \left(\omega(u) - e^{-\gamma}\right) \frac{\mathrm{d}u}{(u+1)^s}.$$

Proof of $d(v) \sim \frac{C}{v+1}$: Inversion of the Laplace transform

Let P_a denote the finite set of poles of $\widehat{G}(s)$ with $-a < \operatorname{Re} s \leq 0$.

$$e^y d(e^y - 1) =: G(y) = \sum_{s_k \in P_a} \operatorname{Res}\left(\widehat{G}(s)e^{ys}; s_k\right) + O_a\left(e^{-ay}\right).$$

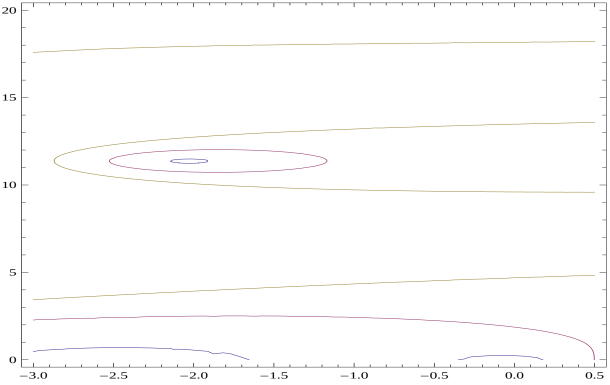

Figure: A contour plot of $|\widehat{G}(s)|$.

Open problems

- For t fixed, is

$$D(x,t) \sim C(t)\,\frac{x\log t}{\log x} \qquad (x \to \infty)$$

for some discontinuous function $C(t)$?

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- Is

$$f(n) \sim \frac{cn}{\log n} \qquad (n \to \infty)$$

  for some constant c, where $f(n)$ is the longest path in the divisor graph ?

- E. Saias, Entiers à diviseurs denses 1, *J. Number Theory* **62** (1997), 163–191.
- E. Saias, Entiers à diviseurs denses 2, *J. Number Theory* **86** (2001), 39–49.
- E. Saias, Applications des entiers à diviseurs denses, *Acta Arith.* **83** (1998), 225–240.
- G. Tenenbaum, Sur un problème de crible et ses applications, *Ann. Sci. École Norm. Sup. (4)* **19** (1986), 1–30.
- G. Tenenbaum, Sur un problème de crible et ses applications, 2. Corrigendum et étude du graphe divisoriel. *Ann. Sci. École Norm. Sup. (4)* **28** (1995), 115–127.
- A. Weingartner, Integers with dense divisors, *J. Number Theory* **108** (2004), 1–17.
- A. Weingartner, Integers with dense divisors 2, *J. Number Theory* **108** (2004), 18–28.