Symmetries of rational functions arising in Ecalle's study of multiple zeta values

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Definition

Definition

The *multiple zeta values* are the numbers

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\cdots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}},$$

for $n_i \in \mathbb{N}$ with $n_r > 1$. Here, r is the *depth* and $n = n_1 + \cdots + n_r$ is the *weight* of $\zeta(n_1, \ldots, n_r)$. Note that $1 \leq r < n$ because $n_r > 1$.

Challenge: Show $\zeta(3) = \zeta(1, 2)$ or $\zeta(4) = 4\zeta(1, 3)$.

Riemann zeta values

When the depth r is 1, the multiple zeta values are just special values of the Riemann zeta function $\zeta(n) = \sum_{0<k} \frac{1}{k^n}$.

$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^n}{2(2n)!} \in \pi^{2n}\mathbb{Q} \text{ (Euler, 1735).}$$

Folklore Conjecture

The numbers π, $\zeta(3)$, $\zeta(5)$, $\zeta(7), \ldots, \zeta(2n+1)$ are algebraically independent over $\mathbb{Q}$ for any integer $n \geq 1$.

Rivoal (2000): Infinitely many of $\zeta(3), \zeta(5), \zeta(7), \ldots$ are irrational.
Zudilin (2001): At least one of $\{\zeta(5), \zeta(7), \zeta(9), \zeta(11)\}$ is irrational.

Multiple zetas

In studying $\zeta(n)$ one is led rapidly to the multiple zeta values. Euler knew that, for all $n, m > 1$, we have

$$\begin{aligned}\zeta(m)\zeta(n) &= \sum_{0<k} \frac{1}{k^m} \sum_{0<\ell} \frac{1}{\ell^n} \\ &= \sum_{0<k<\ell} \frac{1}{k^m \ell^n} + \sum_{0<k=\ell} \frac{1}{k^{m+n}} + \sum_{0<\ell<k} \frac{1}{\ell^n k^m} \\ &= \zeta(m,n) + \zeta(m+n) + \zeta(n,m).\end{aligned}$$

So, for example, $\zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$

More general stuffle relations

For $n_1, n_3 > 1$, we have

$$\begin{aligned}
\zeta(n_1)\zeta(n_2, n_3) &= \sum_{0<k_1} \frac{1}{k_1^{n_1}} \sum_{0<k_2<k_3} \frac{1}{k_2^{n_2} k_3^{n_3}} \\
&= \sum_{0<k_1<k_2<k_3} \frac{1}{k_1^{n_1} k_2^{n_2} k_3^{n_3}} + \sum_{0<k_2<k_1<k_3} \frac{1}{k_2^{n_2} k_1^{n_1} k_3^{n_3}} \\
&+ \sum_{0<k_2<k_3<k_1} \frac{1}{k_2^{n_2} k_3^{n_3} k_1^{n_1}} \text{ (shuffle)} \\
&+ \sum_{0<k_1=k_2<k_3} \frac{1}{k_1^{n_1+n_2} k_3^{n_3}} + \sum_{0<k_2<k_1=k_3} \frac{1}{k_2^{n_2} k_3^{n_1+n_3}} \text{ (+stuff)}
\end{aligned}$$

In general, these relations are called the **stuffle relations**, and give one way of multiplying multiple zeta values and landing back in the space of multiple zeta values!

Multiple zetas as periods

Kontsevich (1992) noticed that multiple zeta values are periods.

$$\mathcal{I}(1,0) = \iint\limits_{0<t_1<t_2<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2} = \iint\limits_{0<t_1<t_2<1} \left(\sum_{0\le n} t_1^n\right) \frac{dt_1\,dt_2}{t_2}$$

$$= \int\limits_{0<t_2<1} \left(\sum_{0\le n} \frac{t_2^{n+1}}{n+1}\right) \frac{dt_1\,dt_2}{t_2} = \int\limits_{0<t_2<1} \left(\sum_{0\le n} \frac{t_2^{n}}{n+1}\right) dt_1\,dt_2$$

$$= \sum_{0\le n} \frac{1}{(n+1)^2} = \zeta(2).$$

Similarly,

$$\mathcal{I}(1,0,0) = \iiint\limits_{0<t_1<t_2<t_3<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{dt_3}{t_3} = \int\limits_{0<t_3<1} \sum_{0\le n} \frac{t_3^n}{(n+1)^2} dt_3$$

$$= \sum_{0\le n} \frac{1}{(n+1)^3} = \zeta(3).$$

Multiple zetas as periods

Finally, with n zeros $\mathcal{I}(1,0,0,\ldots,0)=\zeta(n)$. And we have

$$\begin{aligned}
\mathcal{I}(1,1,0) &= \iiint\limits_{0<t_1<t_2<t_3<1} \frac{dt_1}{1-t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3} \\
&= \iint\limits_{0<t_2<t_3<1} \left(\sum_{0\le n}\frac{t_2^{n+1}}{n+1}\right)\left(\sum_{0\le k} t_2^k\right)\frac{dt_2\,dt_3}{t_3} \\
&= \int\limits_{0<t_3<1} \left(\sum_{0\le n,k}\frac{t_3^{n+k+2}}{(n+1)(n+k+2)}\right)\frac{dt_3}{t_3} \\
&= \sum_{0\le n,k}\frac{1}{(n+1)(n+k+2)^2} \\
&= \sum_{0<m<\ell}\frac{1}{m\ell^2} = \zeta(1,2)
\end{aligned}$$

Shuffle products

- If w is a word in $\{1, 0\}$ that starts in 1 and ends in 0, say $w = 10^{n_1-1}10^{n_2-1}\cdots 10^{n_r-1}$, then $\mathcal{I}(w) = \mathcal{I}(1, 0, 0, \ldots, 1, 0, 0, \ldots\ldots, 1, 0, 0, \ldots) = \zeta(n_1, \ldots, n_r)$.
- If $\mathcal{I}(w_1)$ and $\mathcal{I}(w_2)$ are two such integrals then

$$\mathcal{I}(w_1)\mathcal{I}(w_2) = \sum_{w \in \mathrm{sh}(w_1, w_2)} \mathcal{I}(w),$$

  which gives another way to multiply multiple zeta values.
- Mapping $t_i \to t_{n+1-i}$ and switching t's for $1 - t$'s gives a duality, whereby $\mathcal{I}(w) = \mathcal{I}(\overline{w})$. For example, so $\zeta(3) = \mathcal{I}(1, 0, 0) = \mathcal{I}(1, 1, 0) = \zeta(1, 2)$.

Shuffle product example

$$\begin{aligned}
\zeta(2)^2 = \zeta(2)\zeta(2) &= \iint\limits_{0<t_1<t_2<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \iint\limits_{0<t_1<t_2<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2} \\
&= \int \omega_1\omega_2 \int \omega_3\omega_4 \\
&= \int \omega_1\omega_2\omega_3\omega_4 + \int \omega_1\omega_3\omega_2\omega_4 + \int \omega_1\omega_3\omega_4\omega_2 \\
&+ \int \omega_3\omega_1\omega_2\omega_4 + \int \omega_3\omega_1\omega_4\omega_2 + \int \omega_3\omega_4\omega_1\omega_2 \\
&= 4\int \omega_1^2\omega_2^2 + 2\int \omega_1\omega_2\omega_1\omega_2 \\
&= 4\zeta(1,3) + 2\zeta(2,2)
\end{aligned}$$

Double shuffle relations

Racinet noted that we now have a new family of relations of the form $4\zeta(1,3) + 2\zeta(2,2) = \zeta(2)^2 = 2\zeta(2,2) + \zeta(4)$, from which it follows that $4\zeta(1,3) = \zeta(4)$.

Relations in weight n

- There are a total of 2^{n-2} multiple zeta values of weight n (recall $1 \leq r < n$).
- But the dimension d_n of the $\mathbb{Q}$-vector space of multiple zeta values of a fixed weight n is much, much smaller, owing to the above algebraic relations and perhaps others?
- For example, $28\zeta(3,9) + 150\zeta(5,7) + 168\zeta(7,5) = \frac{5197}{691}\zeta(12)$ (Gangl, Kaneko, Zagier, related to period polynomials for $PSL_2(\mathbb{Z})$)
- We know $d_2 = 1$, and the weight 2 space is generated by $\zeta(2)$. And $d_3 = 1$, and the weight 3 space is generated by $\zeta(1,2) = \zeta(3)$.
- It's not known that $d_n \geq 2$ for any n.

A conjecture of Zagier

Conjecture (Zagier)

The dimension d_n *of the* $\mathbb{Q}$*-vector space of multiple zeta values of weight* n *is given by* $d_1 = 0, d_2 = d_3 = d_4 = 1$*, and* $d_k = d_{k-2} + d_{k-3}$.

J. Ecalle

In order to study these spaces, we define the algebra

Definition

MZV = formal symbols of the form $Z(n_1, \ldots, n_r)$ under the shuffle product modulo the stuffle relations.

This space then is dual to a quotient of a space of rational functions that, when equipped with the appropriate multiplication, is a Lie algebra.

Schneps, working with Ecalle's manuscript, has reduced the proof that this space of rational functions is a Lie algebra to a long complicated combinatorial proof, of which only the last few steps remain.

Symmetries of rational functions

- There are two symmetries of a rational function A that are of interest.
- First, we define $\operatorname{circ} A(u_1, \ldots, u_r) = A(u_r, u_1, \ldots, u_{r-1})$.
- Then A is circ-neutral if $A + \operatorname{circ} A + \operatorname{circ}^2 A + \cdots + \operatorname{circ}^{r-1} A = 0$.
- And A is alternal if $\displaystyle\sum_{w \in sh(u_1 u_2 \ldots u_k)(u_{k+1} \ldots u_r)} A(w) = 0$ for all $1 \leq k \leq r-1$.
- An example of a polynomial that is circ-neutral and alternal is $A(u_1, u_2) = -2u_1^3 - 3u_1^2 u_2 + 3u_1 u_2^2 + 2u_2^3$.

Problem

If A is alternal then A is circ-neutral.

Alternality implies circ-neutrality

Definition

If A and B are polynomials in r_A (resp. r_B) variables then $\text{mu}(A,B)$ is a polynomial in $r = r_A + r_B$ variables defined as

$$\text{mu}(A,B) = A(u_1,\ldots,u_{r_A})B(u_{r_A+1},\ldots,u_{r_A+r_B}).$$

Furthermore, we set $[[A,B]] = \text{mu}(A,B) - \text{mu}(B,A)$.

Note that $\text{mu}(B,A) = B(u_1,\ldots,u_{r_B})A(u_{r_B+1},\ldots,u_{r_A+r_B})$ is also a polynomial in r variables.

Proposition

Let A and B be monomials of degree d_A (resp. d_B) in r_A (resp. r_B) variables. Then $M := [[A,B]]$ is circ-neutral.

So it suffices to show that any alternal polynomial is of the form $[[A,B]]$ for some A and B.

Let $\mathbb{Q}[u_1, u_2, \ldots]$ be the ring of polynomials in commuting variables $u_1, u_2, \ldots$ and let $\mathbb{Q}\langle x, y\rangle$ be the ring of polynomials in two non-commuting variables. Define the map of $\mathbb{Q}$ vector spaces

$$\phi : \mathbb{Q}[u_1, u_2, \ldots] \longrightarrow \mathbb{Q}\langle x, y\rangle \tag{1}$$

by extending linearly from the map

$$u_1^{a_1} \cdots u_r^{a_r} \mapsto C_{a_1+1} C_{a_2+1} \ldots C_{a_{r-1}+1} C_{a_r+1},$$

where

$$C_i = \mathrm{ad}(x)^{i-1} y = [x, \cdots x, [x, [x, y]]] \cdots],$$

and $[x, y] = xy - yx$ is the standard Lie bracket.

Sketch of Proof

1. If B and D are two polynomials then we have
$$\phi(\mathrm{mu}(B, D)) = \phi(B)\phi(D),$$
so that
$$\phi([[B, D]]) = [\phi(B), \phi(D)].$$
2. Alternal polynomials C in $\mathbb{Q}[u_1, u_2, \ldots]$ map to Lie polynomials because a polynomial in $C_1, C_2, \ldots$ that satisfies the shuffle relations is a Lie polynomial.
3. Any Lie polynomial must be of the form $[f, g]$.
4. Thus, it must come from a polynomial of the form $[[A, B]]$, and so must be circ-neutral.

Thanks

Thanks for listening!