# Enumerating Cubic Function Fields

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A classical question is to ask for an estimate of how many fields there are up to a given bound.

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$$\begin{aligned}
\#\left\{\text{squarefree integers} \le X\right\} &\approx X \cdot \tfrac{3}{4} \cdot \tfrac{8}{9} \cdots \\
&\approx X \cdot \left(1 - \tfrac{1}{4}\right) \cdot \left(1 - \tfrac{1}{9}\right) \cdots \\
&\approx X \cdot \prod_{p \text{ prime}} \left(1 - \tfrac{1}{p^2}\right) \\
&\approx \frac{X}{\zeta(2)} \text{ or } \frac{6}{\pi^2} X
\end{aligned}$$

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&= \left(\frac{1}{1^s} + \frac{1}{2^s}\right) \cdot \left(\frac{1}{1^s} + \frac{1}{3^s}\right) \cdot \left(\frac{1}{1^s} + \frac{1}{5^s}\right) \cdots \\
&= \prod_{p \text{ prime}} \left(1 + \tfrac{1}{p^s}\right) = \prod_{p \text{ prime}} \frac{\left(1 - \frac{1}{p^s}\right)^{-1}}{\left(1 - \frac{1}{p^{2s}}\right)^{-1}} = \frac{\zeta(s)}{\zeta(2s)}
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Residue of $\Phi(s)$ at $s = 1$ is $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$!

# Zeta Functions of Function Fields

- Rational Function Field: $k = \mathbb{F}_q(T)$
- Ring of Integers: $\mathcal{O}_k = \mathbb{F}_q[T]$
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$$\zeta_k(s) = \sum_{d=0}^{\infty} \sum_{\deg f = d} \frac{1}{\left|f\right|^s} = 1 + \frac{q}{q^s} + \frac{q^2}{q^{2s}} + \cdots = \frac{1}{1 - q^{1-s}}$$

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In fact, this approximation is an equality for polynomials!

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Cubic Function Fields (Datskovsky & Wright, 1988)

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- $K_2$ is a fixed quadratic extension of base field $K$.
- $\mathcal{F}(K_2)$ is the set of cubic extensions $K_3$ of $K$ with Galois closure $K_6$ containing the quadratic subextension $K_2$.
- $N(X, K_2)$ is the number of non-isomorphic cubic fields $K_3 \in \mathcal{F}(K_2)$ with discriminant bounded by $X$ in absolute value.

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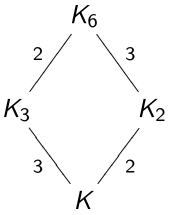


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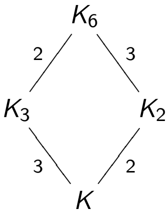


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Cubic Function Fields

$$N(q^{2n}, K_2) = \ ?$$

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The conductor $\mathfrak{f}(K_3/K)$ is (roughly) a measure of the amount of ramification in $K_3/K$ that doesn't come from $K_2/K$.

# Fundamental Dirichlet Series

Our goal is to find the residue at $s = 1$ of the Dirichlet Series

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In order to do this, we need a classification of isomorphism classes of cubic function fields with quadratic resolvent $K_2$.

In the process, we will convert conductors of $K_3$ into ideals of $K_2$.

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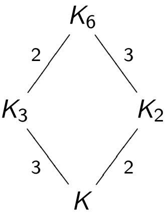


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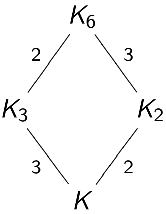


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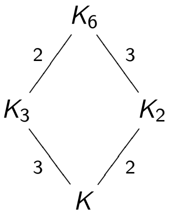


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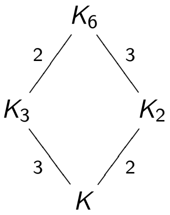


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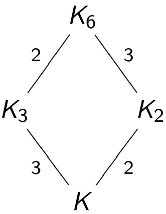


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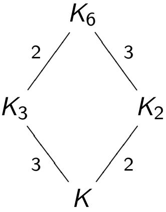
Kummer Theory: $K_6 = K_2(\sqrt[3]{\alpha})$ for some $\alpha \in K_2^\times \setminus (K_2^\times)^3$.

# The Galois Group of $K_6$

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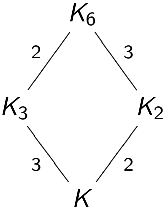


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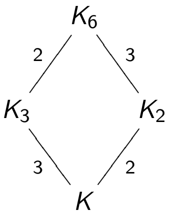


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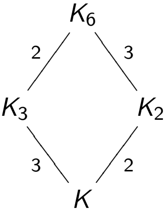


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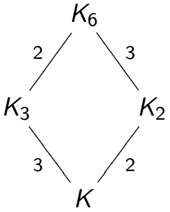


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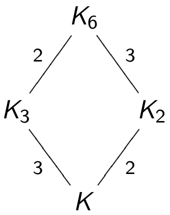


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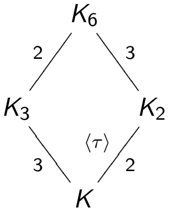
Equivalent to all (non-cubed) primes of $K$ lying below $(\alpha)$ to split in $K_2$.

# Isomorphism Classes of Cubic Function Fields




In $K_2$,

$$(\alpha) = \mathfrak{a}_0 \mathfrak{a}_1^2 \mathfrak{q}^3$$

where $\mathfrak{a}_0$, $\mathfrak{a}_1$ are coprime, squarefree ideals such that $\overline{\mathfrak{a}_0 \mathfrak{a}_1^2} \in \mathrm{Cl}(K_2)^3$ and $\mathfrak{a}_1 = \tau(\mathfrak{a}_0)$.

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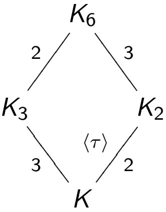


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Every (non-cubed) prime $p_i$ of $K$ lying below $\alpha$ splits in $K_2$:

$$p_i = \mathfrak{p}_i\tau(\mathfrak{p}_i), \quad \mathfrak{a}_0 = \mathfrak{p}_1\cdots\mathfrak{p}_k, \quad \mathfrak{a}_1 = \tau(\mathfrak{p}_1)\cdots\tau(\mathfrak{p}_k)$$

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**Theorem**

*There exists a bijection between isomorphism classes of cubic extensions $K_3/K$ with given quadratic resolvent $K_2$ and triples $(\mathfrak{a}_0, \mathfrak{a}_1, \overline{u})$ under the equivalence $(\mathfrak{a}_0, \mathfrak{a}_1, \overline{u}) \sim (\mathfrak{a}_1, \mathfrak{a}_0, 1/\overline{u})$, where $\mathfrak{a}_0$, $\mathfrak{a}_1$ are as above and $\overline{u}$ is an element of the 3-Selmer group lying in the kernel of the norm map.*

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# Fundamental Dirichlet Series

After a LOT of manipulation,

$$\Phi(s, K_2) = c_1 \prod_{p \in \mathcal{D}} \left(1 + \frac{2}{|\mathbf{N}p|^s}\right) + O(1)$$

where $\mathcal{D}$ be the primes $p$ of $K$ that split in $K_2$ and $c_1$ is a constant depending on the unit group of $K_2$.

# Fundamental Dirichlet Series

After MORE simplifying,

$$\Phi(s, K_2) = c_1 \prod_{p \in \mathcal{D}} \left(1 + \frac{2}{\left|\mathbf{N}p\right|^s}\right) \cdot \frac{\zeta_{K_2}(s)}{\zeta_K(2s) \cdot \frac{\zeta_K(s)}{\zeta_K(2s)} \cdot L(s, \psi)} + O(1)$$

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$$\operatorname{Res}_{s=1} \Phi(s, K_2) = \frac{c_1}{\zeta_K(2)} \cdot \prod_{p \in \mathcal{D}} \left(1 - \frac{2}{|\mathbf{N}p|(|\mathbf{N}p| + 1)}\right) \cdot \prod_{p | \Delta} \left(1 - \frac{1}{|\mathbf{N}p| + 1}\right) \cdot \operatorname{Res}_{s=1} \zeta_{K_2}(s)$$

# Asymptotic

This residue become the constant term in front of the main term.

Let $N(q^{2n}, K_2)$ be the number of cubic function fields with given quadratic resolvent whose discriminant is bounded by $q^{2n}$ in absolute value. Then

$$N(q^{2n}, K_2) = \mathrm{Res}_{s=1}\Phi(s, K_2) \cdot q^{2n} + O(q^{4n/3+\epsilon})$$

# Conclusions

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- In function fields, some zeta functions and infinite products are rational functions in $q^s$. Hoping to use special cases to investigate closed forms for these terms.
- Error bounds? Progress for number fields (Bhargava et. al., Tamaguchi & Thorne), less for function fields.