Something New in Diophantine Equations

Eva Goedhart

Bryn Mawr College

History

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has no solution in integers with x, y, $z > 1$ *and* $a^2x^k \neq b^2y^k$.

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Let n and N are positive integers with $n \geq 3$ and $\mu_n = \prod_{\substack{p|n \\ p \text{ prime}}} p^{1/(p-1)}$.

If $\left(\sqrt{N} + \sqrt{N+1}\right)^{2(n-2)} > (n\mu_n)^n$, then for any p, $q \in \mathbb{Z}^+$,

$$\left| \sqrt[n]{1 + \frac{1}{N}} - \frac{p}{q} \right| > \frac{1}{8n\mu_n N q^\lambda}$$

where $\lambda = 1 + \dfrac{\log\left(\left(\sqrt{N} + \sqrt{N+1}\right)^2 n\mu_n\right)}{\log\left(\left(\sqrt{N} + \sqrt{N+1}\right)^2 / n\mu_n\right)}$.

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For $n = k$, $N = uv^2$, apply Bennett to $\alpha = \sqrt[k]{1 + \frac{1}{uv^2}}$ and $\beta = \frac{xy}{z^2}$.

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For $k = 9$, $\lambda < \Lambda_9(2^9) < 3.2$, then $42 < 3$

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For $k = 7$ and 8, similar reasoning $\Rightarrow\Leftarrow$, EXCEPT for $(k, a^2cx^k) \in \mathcal{S}$,

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Theorem (G-)

Let a, b, c, $k \in \mathbb{Z}^+$ with $k \geq 7$. The equation

$$(a^2cx^k - 1)(b^2cy^k - 1) = (abcz^k - 1)^2$$

has no solution in integers with x, y, $z > 1$ and $a^2x^k \neq b^2y^k$.