COEFFICIENT CONVERGENCE OF RECURSIVELY DEFINED POLYNOMIALS

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REFERENCE

Limits of Polynomial Sequences
Clark Kimberling
Fibonacci Quarterly, 50.4
2012, pp. 294 - 297

Outline of Talk

- Review paper (What is known)
- State open problem from paper
- Partially solve open problem

EXAMPLE FROM KIMBERLING'S ARTICLE

$$G_n(X) = -XG_{n-1}(X) + (X^2 + 2X)G_{n-2}(X) + X + 1$$

GENERALIZATION

$$G_n(X) = (aX + b)G_{n-1}(X) + (cX^2 + dX + e)G_{n-2}(X) + fX + g$$

RESTRICTIONS

$a \neq 0, \qquad b = 0, \qquad e = 0$

GOAL: Describe the limit

NUMERICAL VALUES

G(X)	Constant	Coef X	Coef X^2	Coef X^3	Coef X^4	Coef X^5	Coef X^6
G_0(X)	1						
G_1(X)	1	1					
G_2(X)	1	2					
G_3(X)	1	2	1	1			
G_4(X)	1	2	3	1	-1		
G_5(X)	1	2	3	1	2	2	
G_6(X)	1	2	3	5	4	-3	-3
G_7(X)	1	2	3	0	5	1	9
G_8(X)	1	2	3	5	8	13	-3

CONVERGENCE IN TWO SENSES

Coefficient Convergence

$$g_i^{(n)} \longrightarrow F_{i+2}, \qquad \text{with } G_n(X) = \sum_{i=0}^{\infty} g_i^{(n)} X^i$$

Pointwise convergence

$$\sum_{i=0}^{\infty} F_{i+2} X^i = \frac{1+X}{1-X-X^2}$$

GENERAL THEOREM

Assumptions

$$G_n(X) = (aX+b)G_{n-1}(X)+(cX^2+dX+e)G_{n-2}(X)+fX+g, \qquad a \neq 0, e = 0, b = 0$$

Theorem

$$G_n(X) = \frac{g+fX}{1-(a+d)X-cX^2}$$

KIMBERLING'S OPEN PROBLEMS

- Is there some type of convergence without the assumptions on *a, e, b*
- Does the result generalize to recursions of order $m > 2$ (The above example is of order 2)

APPROACH OF THIS PRESENTATION

- Focus on coefficient convergence; ignore pointwise convergence
- Main result: $g^{(i)}_n$ *eventually* deg(*i*)-polynomial in *n.*
- Can generalize to higher order recursions
- Can tell you information about the deg(*i*) polynomial.

THEOREM ASSUMPTIONS: DEGREE 2

$$G_n(X) = p_1(X)G_{n-1}(X) + p_2(X)G_{n-2}(X)$$

With initial conditions $G_0(X) = 1, \qquad G_1(X) = 1 + x$

With $p_1(X) = a + bX, \qquad p_2 = c + dX + eX^2, \qquad p_0 = 0$

RESTRICTIONS: *a*=1, *c*=0

Assumption Comparison	Kimberling	This paper
p_0(X)	Non zero	zero
Constant in p_1(X)	zero	Must be 1
Coef of X in p_1(X)	Not zero	Don't care
Constant in p_2(X)	zero	zero

Conclusion Comparison	Kimberling	This Paper
Coefficient convergence	To a constant	To a polynomial value
Pointwise convergence	Yes	No
Coefficient Sequence	Recursive sequence	Patterns in difference triangle

THEOREM ASSUMPTIONS: GENERAL CASE

$$G_n(X) = \sum_{i=1}^{m} p_i(X) G_{n-i}(X)$$

With initial conditions

$$G_i(X) = \sum_{j=0}^{i} X^j, \qquad 0 \le i \le m-1$$

With

$$p_i(X) = \sum_{j=0}^{i} c_j^{(i)} X^j, \qquad 1 \le i \le m, \qquad \text{no constant polynomial}$$

RESTRICTIONS:

$$c_0^{(1)} = 1, \qquad c_0^{(m)} = 0, \qquad m > 1$$

EXAMPLE: $p_1(X) = 1 - 2X, \qquad p_2(X) = X - X^2$

n = \ $G_n(X)$	Constant	Coef X	Coef X^2	Coef X^3	Coef X^4
1	1				
2	1	1			
3	1	0	-3		
4	1	-1	-3	5	
5	1	-2	-2	8	-7
6	1	-3	0	10	-15
7	1	-4	3	10	-25
8	1	-5	7	7	-35
9	1	-6	12	0	-42
10	1	-7	18	-12	-42
11	1	-8	25	-30	-30
12	1	-9	33	-55	0

THEOREM RESULTS

- 4 Results
  - *Diagonal*
  - *Left*
  - *Column Degree*
  - *Triangular Shape*
- **CORROLLARY**: $g^{(i)}_n$ is *eventually* a deg(i) polynomial in n

THEOREM RESULTS – $g_r^{(c)}, \qquad r \geq 0, c \geq 0$

- Diagonal $D_i = g_i^{(i)}, \qquad D_i = bD_{i-1} + dD_{i-2}$
- Left $g_n^{(0)} = 1, \qquad n \geq 0$
- Column Degree $\Delta^i g_n^{(i)} = (b+d)^i \longrightarrow \deg g_n^{(i)} = i$
- For General Case (order m): $\Delta^i g_n^{(i)} = \left(\sum_{j=1}^{m} c_j^{(1)}\right)^i$
- Triangular Support $g_n^{(i)} \neq 0 \longrightarrow 0 \leq i \leq n < \infty$

PROOF OF LEFT (Straightforward)

$G_0(X) = 1, G_1(X) = 1 + X, G_n(X) = (1 + bX)G_{n-1} + (cX + dX^2)G_{n-2}$

Hence $g_n^{(0)} = 1, \qquad n \geq 0$

PROOF OF TRIANGLE (Straightforward)

By induction
True for top and 2^{nd} *row by initial conditions*
True for n-th row by defining recursion

PROOF OF DIAGONAL (Straightforward)

Compare coefficients of degree *n* in defining recursion

$G_n(X) = (a + bX)G_{n-1}(X) + (cX + dX^2)G_{n-2}(X)$

$D_n = bD_{n-1} + dD_{n-2}$

PROOF OF COLUMN DEGREE (Order 2)

$$G_n(X) = (a + bX)G_{n-1}(X) + (cX + dX^2)G_{n-2}, \qquad a = 1$$

$g_n^{(i)} = g_{n-1}^{(i)} + bg_{n-1}^{(i-1)} + cg_{n-2}^{(i-1)} + dg_{n-2}^{(i-2)}$

KEY TRICK

$\Delta g_{n-1}^{(i)} = bg_{n-1}^{(i-1)} + cg_{n-2}^{(i-1)} + dg_{n-2}^{(i-2)} \longrightarrow$

$$\begin{aligned}\Delta^i g_{n-1}^{(i)} &= b\Delta^{i-1} g_{n-1}^{(i-1)} + c\Delta^{i-1} g_{n-2}^{(i-1)} + d\Delta^{i-1} g_{n-2}^{(i-2)} \\ &= b(b+d)^{i-1} + d(b+d)^{i-1} \\ &= (b+d)^i\end{aligned}$$

GOOD EXAMPLE

$G_0(X) = 1, \qquad G_1(X) = 1 + X$

$p_1(X) = 1 - 2X, \qquad p_2(X) = X - X^2, p_0(X) = 0$

$G_n(X) = p_1(X)G_{n-1}(X) + p_2(X)G_{n-2}(X)$

Theorem says that the $G_n(X)$ converge. But to what?

NUMERICAL DATA – PATTERNS in $g^{(c)}_r$ (OEIS: A252840)

G_n(X)	Constant	Coef X	Coef X^2	Coef X^3	Coef X^4
G_0(X)	1				
G_1(X)	1	1			
G_2(X)	1	0	-3		
G_3(X)	1	-1	-3	5	
G_4(X)	1	-2	-2	8	-7
G_5(X)	1	-3	0	10	-15
G_6(X)	1	-4	3	10	-25
G_7(X)	1	-5	7	7	-35
G_8(X)	1	-6	12	0	-42
G_9(X)	1	-7	18	-12	-42
G_{10}(X)	1	-8	25	-30	-30
G_{11}(X)	1	-9	33	-55	0

CLOSED FORM FOR COEFFICIENTS AND $G_n(X)$

Acknowledgement to Robert Israel (OEIS Editor) and David Thornton for helpful conversations

- $G_n(X) = (1+2X)(1-X)^n - 2X(-X)^n$
- $\longrightarrow g_n^{(i)} = (-1)^i \frac{1}{i!}(n)_{i-1}(n-(3i-1))$
- So no coefficient convergence: Coefficients blow up at infinity
- No pointwise convergence: limit function has discontinuities and diverges
- Main point: Although G_{oo} does not exist, the defining recursion gives rise to interesting patterns in coefficients

ILLUSTRATION OF MAIN THEOREM

i) Left most column is all ones
ii) Right most diagonal are odds with alternating signs
iii) Right most diagonal (odds) satisfies recursion: $D_i = -2D_{i-1} - D_{i-2}$
iv) Coefficient triangle satisfies stronger condition: $\Delta g_n^{(i+1)} = -g_n^{(i)} \longrightarrow \Delta^i g_n^{(i)} = (-1)^i$
Proof of (iv) below

IDEA OF PROOF

- Boundary conditions
  - Left most column identically 1
  - Diagonal are odds
- Must prove $\Delta g_{n-1}^{(i)} = -g_{n-1}^{(i-1)}$

REDUCED TO PROVING

$\Delta g_{n-1}^{(i)} = -g_{n-1}^{(i-1)}$

With

$g_n^{(i)} = -(-1)^i \frac{1}{i!}(n - c_i)(n)_{(i-1)}, \qquad c_i = 3i - 1$

Proof of identity in polynomials in two variables!

(TEASE FOR GRADUATE STUDENTS) METHODS OF PROOF

1) Proof: By straightforward manipulations
2) Proof: Clear
3) Proof: Verifiable on Mathematica or any equivalent algebraic software package
4) None of the above

OUTLINE OF PROOF

Write out identity to be proved

Make cancellations

What is left turns out to be quadratic identity in two variables

To prove a quadratic identity we only need 3 cleverly selected points

WHAT HAS TO BE PROVED

$g_{n+1}^{(i+1)} - g_n^{(i+1)} = -g_n^{(i)}$

$g_n^{(i)} = -(-1)^i \frac{1}{i!}(n - c_i)(n)_{(i-1)}, \qquad c_i = 3i - 1$

WHAT CANCELS

- **Minus sign**: On both sides cancel
- **Absolute Factorials**: Factor of (i+1) on right side
- **Falling factorials**: What is common to $(n+1)_i, \qquad (n)_i, \qquad (n)_{i-1}$?
- **Falling factorials continued**: $(n)_{i-1}$ is common
- **Falling factorials continued again**: So what is left
    - (n+1) in first summand on left side
    - n-(i-1) in second summand on left side
    - 1 on right side

WHAT IS LEFT TO PROVE AFTER CANCELLATIONS

$(n+1)(n+1-c_{i+1}) - (n-(i-1))(n-c_{i+1}) = (i+1)(n-c_i)$

Quadratic polynomial identity in *n, i*

Need three clever points to identify

- $n = c_{i+1} - 1$
- $n = c_{i+1}$
- $n = c_i$

CASE: n=c_m

Need to prove $(n+1)(n+1-c_{i+1}) = (n-(i-1))(n-c_{i+1})$

$(c_i+1)(c_i+1-c_{i+1}) = (c_i-(i-1))(c_i-c_{i+1}), \qquad c_i = 3i-1$

$(c_i+1)(-2) = (c_i-(i-1))(-3)$

$-2 \times 3i = -3 \times 2i$

ANOTHER TEASE FOR GRADUATE STUDENTS

Other cases proven similarly.

IS EXAMPLE UNIQUE?

Acknowledgement to Bart Goddard for raising this question

- In general if $\Delta^i g_n^{(i)} = (-1)^i$, we do not necessarily have $\Delta g_n^{(i)} = -g_n^{(i-1)}$
- For example: $G_n(X) = (1+X)G_{n-1}(X) + (X^2-2X)G_{n-2}$ has $\Delta^i g_n^{(i)} = (-1)^n, \Delta g_n^{(i)} \neq -g_n^{(i-1)}$
- However there is a one parameter family of examples. We state without proof the following:
- Theorem: If *e=1+b, d=-e,* and $G_n(X) = (1+bX)G_{n-1}(X) + (dX+eX^2)G_{n-2}(X)$
- Then: $\Delta g_n^{(i)} = -g_n^{(i-1)}$
- The example we gave above illustrates *b=-2.*
- We can actually replace “If” with “If and only if” in the Theorem statement.