The p-adic Arakawa-Kaneko zeta functions and p-adic Lerch transcendent

Paul Thomas Young

College of Charleston

December 17, 2014

Arakawa-Kaneko zeta functions

For a positive integer k, the Arakawa-Kaneko zeta function $\xi_k(s,a)$ is defined for $\Re(s) > 0$ and $\Re(a) > 0$ by the Mellin transform integral

$$\Gamma(s)\xi_k(s,a) = \int_0^\infty t^{s-1} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-at}\, dt \qquad \text{where} \qquad \mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

denotes the order k polylogarithm function.

Arakawa-Kaneko zeta functions

For a positive integer k, the Arakawa-Kaneko zeta function $\xi_k(s,a)$ is defined for $\Re(s) > 0$ and $\Re(a) > 0$ by the Mellin transform integral

$$\Gamma(s)\xi_k(s,a) = \int_0^\infty t^{s-1}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{-at}\,dt \qquad \text{where} \qquad \mathrm{Li}_k(z) = \sum_{m=1}^\infty \frac{z^m}{m^k}$$

denotes the order k polylogarithm function.

- When $k = 1$ we have $\xi_1(s,a) = s\zeta(s+1,a)$ and when $s = a = 1$ we have $\xi_k(1,1) = \zeta(k+1)$ in terms of the Hurwitz zeta and Riemann zeta functions.

Arakawa-Kaneko zeta functions

For a positive integer k, the Arakawa-Kaneko zeta function $\xi_k(s,a)$ is defined for $\Re(s) > 0$ and $\Re(a) > 0$ by the Mellin transform integral

$$\Gamma(s)\xi_k(s,a) = \int_0^\infty t^{s-1} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-at}\,dt \qquad \text{where} \qquad \mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

denotes the order k polylogarithm function.

- When $k = 1$ we have $\xi_1(s,a) = s\zeta(s+1,a)$ and when $s = a = 1$ we have $\xi_k(1,1) = \zeta(k+1)$ in terms of the Hurwitz zeta and Riemann zeta functions.

- More generally the values

$$\xi_{k-1}(m,1) = \zeta^\star(k,\underbrace{1,\ldots,1}_{m-1}) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1^k n_2 \cdots n_m}$$

  are *non-strict multiple zeta values* (also called *multiple zeta-star values*), whose arithmetic has been extensively studied.

Arakawa-Kaneko zeta functions

For a positive integer k, the Arakawa-Kaneko zeta function $\xi_k(s,a)$ is defined for $\Re(s) > 0$ and $\Re(a) > 0$ by the Mellin transform integral

$$\Gamma(s)\xi_k(s,a) = \int_0^\infty t^{s-1} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-at}\,dt \qquad \text{where} \qquad \mathrm{Li}_k(z) = \sum_{m=1}^\infty \frac{z^m}{m^k}$$

denotes the order k polylogarithm function.

- When $k=1$ we have $\xi_1(s,a) = s\zeta(s+1,a)$ and when $s=a=1$ we have $\xi_k(1,1) = \zeta(k+1)$ in terms of the Hurwitz zeta and Riemann zeta functions.

- More generally the values

$$\xi_{k-1}(m,1) = \zeta^\star(k, \underbrace{1,\dots,1}_{m-1}) := \sum_{n_1 \ge n_2 \ge \cdots \ge n_m \ge 1} \frac{1}{n_1^k n_2 \cdots n_m}$$

  are *non-strict multiple zeta values* (also called *multiple zeta-star values*), whose arithmetic has been extensively studied.

- The values $\xi_k(m,a)$ may also be expressed as series of generalized harmonic number sums. Our objective is to describe p-adic analogues of these zeta functions and interpret these harmonic number series expressions p-adically.

Multiplicative Structure of $\mathbb{C}_p^\times$

Let $\mathbb{C}_p$ denote the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers.

- Nonzero complex numbers are often represented in polar coordinates. As multiplicative groups we have $\mathbb{C}^\times \cong \mathbb{R}^+ \times S^1$ given by $z = re^{i\theta} \mapsto (r, e^{i\theta})$.

Multiplicative Structure of $\mathbb{C}_p^\times$

Let $\mathbb{C}_p$ denote the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers.

- Nonzero complex numbers are often represented in polar coordinates. As multiplicative groups we have $\mathbb{C}^\times \cong \mathbb{R}^+ \times S^1$ given by $z = re^{i\theta} \mapsto (r, e^{i\theta})$.

- Similarly there is an internal direct product decomposition of multiplicative groups $\mathbb{C}_p^\times \cong p^{\mathbb{Q}} \times \mu \times D$ given by $z = p^{\nu_p(z)} \cdot \hat{z} \cdot \langle z \rangle \mapsto (p^{\nu_p(z)}, \hat{z}, \langle z \rangle)$,

  where $D = \{z \in \mathbb{C}_p : |z-1|_p < 1\}$ and μ is the group of roots of unity of order not divisible by p. The unique element $\hat{z}$ of μ closest to $z/p^{\nu(z)}$ is called the *Teichmüller representative* of z, and can be defined analytically by $\hat{z} = \lim_{n\to\infty}(z/p^{\nu_p(z)})^{p^{n!}}$.

Multiplicative Structure of $\mathbb{C}_p^\times$

Let $\mathbb{C}_p$ denote the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers.

- Nonzero complex numbers are often represented in polar coordinates. As multiplicative groups we have $\mathbb{C}^\times \cong \mathbb{R}^+ \times S^1$ given by $z = re^{i\theta} \mapsto (r, e^{i\theta})$.
- Similarly there is an internal direct product decomposition of multiplicative groups $\mathbb{C}_p^\times \cong p^{\mathbb{Q}} \times \mu \times D$ given by $z = p^{\nu_p(z)} \cdot \hat{z} \cdot \langle z \rangle \mapsto (p^{\nu_p(z)}, \hat{z}, \langle z \rangle)$,

  where $D = \{z \in \mathbb{C}_p : |z-1|_p < 1\}$ and μ is the group of roots of unity of order not divisible by p. The unique element $\hat{z}$ of μ closest to $z/p^{\nu(z)}$ is called the *Teichmüller representative* of z, and can be defined analytically by $\hat{z} = \lim_{n\to\infty}(z/p^{\nu_p(z)})^{p^{n!}}$.
- Note that the disc D is a multiplicative group (!) and that $\langle z \rangle = z/(p^{\nu_p(z)}\hat{z})$ is the projection of z onto D. The elements of D are sometimes called "positive" elements of $\mathbb{C}_p$.

  The projection $z \mapsto \langle z \rangle$ has derivative $\frac{d}{dz}\langle z \rangle = \langle z \rangle / z$; the expression $\langle z \rangle / z$ is locally constant on $\mathbb{C}_p^\times$.

Multiplicative Structure of $\mathbb{C}_p^\times$

Let $\mathbb{C}_p$ denote the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers.

- Nonzero complex numbers are often represented in polar coordinates. As multiplicative groups we have $\mathbb{C}^\times \cong \mathbb{R}^+ \times S^1$ given by $z = re^{i\theta} \mapsto (r, e^{i\theta})$.
- Similarly there is an internal direct product decomposition of multiplicative groups $\mathbb{C}_p^\times \cong p^{\mathbb{Q}} \times \mu \times D$ given by $z = p^{\nu_p(z)} \cdot \hat{z} \cdot \langle z \rangle \mapsto (p^{\nu_p(z)}, \hat{z}, \langle z \rangle)$,

  where $D = \{z \in \mathbb{C}_p : |z-1|_p < 1\}$ and μ is the group of roots of unity of order not divisible by p. The unique element $\hat{z}$ of μ closest to $z/p^{\nu(z)}$ is called the *Teichmüller representative* of z, and can be defined analytically by $\hat{z} = \lim_{n\to\infty}(z/p^{\nu_p(z)})^{p^{n!}}$.
- Note that the disc D is a multiplicative group (!) and that $\langle z \rangle = z/(p^{\nu_p(z)}\hat{z})$ is the projection of z onto D. The elements of D are sometimes called "positive" elements of $\mathbb{C}_p$.

  The projection $z \mapsto \langle z \rangle$ has derivative $\frac{d}{dz}\langle z \rangle = \langle z \rangle / z$; the expression $\langle z \rangle / z$ is locally constant on $\mathbb{C}_p^\times$.
- This decomposition of $\mathbb{C}_p^\times$ depends on our choice of embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$; the projections $p^{\nu_p(z)}, \hat{z}, \langle z \rangle$ are only determined up to roots of unity. However for $z \in \mathbb{Q}_p^\times$ the projections $p^{\nu_p(z)}, \hat{z}, \langle z \rangle$ are uniquely determined and do not depend on the choice of embedding. We choose such an embedding and fix it once and for all.

p-adic exp, log, and power functions

- The exponential function is not p-adically entire; $\exp_p(z) = \sum_{n=0}^{\infty} z^n/n!$ converges only on $\{z \in \mathbb{C}_p : |z|_p < p^{-1/(p-1)}\}$.

p-adic exp, log, and power functions

- The exponential function is not p-adically entire; $\exp_p(z) = \sum_{n=0}^{\infty} z^n/n!$ converges only on $\{z \in \mathbb{C}_p : |z|_p < p^{-1/(p-1)}\}$.

- But $\log_p(z) = -\sum_{n=1}^{\infty} (1-z)^n/n$ converges on $D = \{z \in \mathbb{C}_p : |z-1|_p < 1\}$.

  We extend $\log_p$ to a function on all of $\mathbb{C}_p^\times$ by setting $\log_p z = \log_p \langle z \rangle$, where $z = p^{\nu(z)} \cdot \hat{z} \cdot \langle z \rangle$. Then $\log_p(wz) = \log_p w + \log_p z$ for all $w, z \in \mathbb{C}_p^\times$.

  The derivative $\frac{d}{dz} \log_p z = 1/z$ for all $z \in \mathbb{C}_p^\times$. The quantity $\langle z \rangle / z$ is always algebraic and has logarithm zero for all $z \in \mathbb{C}_p^\times$.

p-adic exp, log, and power functions

- The exponential function is not p-adically entire; $\exp_p(z) = \sum_{n=0}^{\infty} z^n/n!$ converges only on $\{z \in \mathbb{C}_p : |z|_p < p^{-1/(p-1)}\}$.

- But $\log_p(z) = -\sum_{n=1}^{\infty} (1-z)^n/n$ converges on $D = \{z \in \mathbb{C}_p : |z-1|_p < 1\}$.

  We extend $\log_p$ to a function on all of $\mathbb{C}_p^\times$ by setting $\log_p z = \log_p \langle z \rangle$, where $z = p^{\nu(z)} \cdot \hat{z} \cdot \langle z \rangle$. Then $\log_p(wz) = \log_p w + \log_p z$ for all $w, z \in \mathbb{C}_p^\times$.

  The derivative $\frac{d}{dz} \log_p z = 1/z$ for all $z \in \mathbb{C}_p^\times$. The quantity $\langle z \rangle / z$ is always algebraic and has logarithm zero for all $z \in \mathbb{C}_p^\times$.

- We define $z^s = \sum_{n=0}^{\infty} \binom{s}{n} (z-1)^n$ whenever the series converges.

- The exponential function is not p-adically entire; $\exp_p(z) = \sum_{n=0}^{\infty} z^n/n!$ converges only on $\{z \in \mathbb{C}_p : |z|_p < p^{-1/(p-1)}\}$.

- But $\log_p(z) = -\sum_{n=1}^{\infty}(1-z)^n/n$ converges on $D = \{z \in \mathbb{C}_p : |z-1|_p < 1\}$.

  We extend $\log_p$ to a function on all of $\mathbb{C}_p^{\times}$ by setting $\log_p z = \log_p \langle z \rangle$, where $z = p^{\nu(z)} \cdot \hat{z} \cdot \langle z \rangle$. Then $\log_p(wz) = \log_p w + \log_p z$ for all $w, z \in \mathbb{C}_p^{\times}$.

  The derivative $\frac{d}{dz} \log_p z = 1/z$ for all $z \in \mathbb{C}_p^{\times}$. The quantity $\langle z \rangle / z$ is always algebraic and has logarithm zero for all $z \in \mathbb{C}_p^{\times}$.

- We define $z^s = \sum_{n=0}^{\infty} \binom{s}{n}(z-1)^n$ whenever the series converges.

- For any $z \in \mathbb{C}_p^{\times}$, the function $s \mapsto \langle z \rangle^s$ is a C^{∞} function of s on $\mathbb{Z}_p$, and analytic on a disc of positive radius about $s = 0$, and on this disc $\langle z \rangle^s = \exp_p(s \log_p \langle z \rangle) = \exp_p(s \log_p z)$.

p-adic exp, log, and power functions

- The exponential function is not p-adically entire; $\exp_p(z) = \sum_{n=0}^{\infty} z^n/n!$ converges only on $\{z \in \mathbb{C}_p : |z|_p < p^{-1/(p-1)}\}$.

- But $\log_p(z) = -\sum_{n=1}^{\infty}(1-z)^n/n$ converges on $D = \{z \in \mathbb{C}_p : |z-1|_p < 1\}$.

  We extend $\log_p$ to a function on all of $\mathbb{C}_p^{\times}$ by setting $\log_p z = \log_p \langle z \rangle$, where $z = p^{\nu(z)} \cdot \hat{z} \cdot \langle z \rangle$. Then $\log_p(wz) = \log_p w + \log_p z$ for all $w, z \in \mathbb{C}_p^{\times}$.

  The derivative $\frac{d}{dz} \log_p z = 1/z$ for all $z \in \mathbb{C}_p^{\times}$. The quantity $\langle z \rangle / z$ is always algebraic and has logarithm zero for all $z \in \mathbb{C}_p^{\times}$.

- We define $z^s = \sum_{n=0}^{\infty} \binom{s}{n}(z-1)^n$ whenever the series converges.

- For any $z \in \mathbb{C}_p^{\times}$, the function $s \mapsto \langle z \rangle^s$ is a C^{∞} function of s on $\mathbb{Z}_p$, and analytic on a disc of positive radius about $s = 0$, and on this disc $\langle z \rangle^s = \exp_p(s \log_p \langle z \rangle) = \exp_p(s \log_p z)$.

- Also, for any $s \in \mathbb{Z}_p$, the function $z \mapsto \langle z \rangle^s$ is a locally analytic function of z on $\mathbb{C}_p^{\times}$.

  We have $\frac{d}{dz}\langle z \rangle^s = s\langle z \rangle^s/z$ and $\frac{d}{ds}\langle z \rangle^s = \langle z \rangle^s \log_p z$.

Barnes zeta functions of arbitrary integer order

Consider the complex and p-adic multiple zeta functions of order $r \in \mathbb{Z}^+$ defined by

$$\zeta_r(s,a) = \sum_{\bar{t} \in \mathbb{Z}_0^r} (a + |\bar{t}|)^{-s}, \qquad \zeta_{p,r}(s,a) = \frac{1}{(s-1)\cdots(s-r)} \int_{\mathbb{Z}_p^r} \frac{(a+|\bar{t}|)^r}{\langle a+|\bar{t}|\rangle^s}\, d\bar{t},$$

where $|\bar{t}| = t_1 + \cdots + t_r$ denotes the "length" of the vector $\bar{t} = (t_1, ..., t_r)$.

Consider the complex and p-adic multiple zeta functions of order $r \in \mathbb{Z}^+$ defined by

$$\zeta_r(s,a) = \sum_{\bar{t} \in \mathbb{Z}_0^r} (a + |\bar{t}|)^{-s}, \qquad \zeta_{p,r}(s,a) = \frac{1}{(s-1)\cdots(s-r)} \int_{\mathbb{Z}_p^r} \frac{(a+|\bar{t}|)^r}{\langle a+|\bar{t}|\rangle^s}\, d\bar{t},$$

where $|\bar{t}| = t_1 + \cdots + t_r$ denotes the "length" of the vector $\bar{t} = (t_1, ..., t_r)$.

- When $r = 0$ these are just the functions $\zeta_0(s,a) = a^{-s}$ and $\zeta_{p,0}(s,a) = \langle a \rangle^{-s}$, and the difference equation

$$\zeta_{p,r}(s,a) - \zeta_{p,r}(s,a+1) = -\zeta_{p,r-1}(s,a)$$

is satisfied by both $\zeta_r(s,a)$ and $\zeta_{p,r}(s,a)$.

Barnes zeta functions of arbitrary integer order

Consider the complex and p-adic multiple zeta functions of order $r \in \mathbb{Z}^+$ defined by

$$\zeta_r(s,a) = \sum_{\bar{t} \in \mathbb{Z}_0^r} (a + |\bar{t}|)^{-s}, \qquad \zeta_{p,r}(s,a) = \frac{1}{(s-1)\cdots(s-r)} \int_{\mathbb{Z}_p^r} \frac{(a+|\bar{t}|)^r}{\langle a + |\bar{t}| \rangle^s} \, d\bar{t},$$

where $|\bar{t}| = t_1 + \cdots + t_r$ denotes the "length" of the vector $\bar{t} = (t_1, ..., t_r)$.

- When $r = 0$ these are just the functions $\zeta_0(s,a) = a^{-s}$ and $\zeta_{p,0}(s,a) = \langle a \rangle^{-s}$, and the difference equation
$$\zeta_{p,r}(s,a) - \zeta_{p,r}(s,a+1) = -\zeta_{p,r-1}(s,a)$$
is satisfied by both $\zeta_r(s,a)$ and $\zeta_{p,r}(s,a)$.
- Rewriting the difference equations
$$\zeta_{p,-r}(s,a) = \zeta_{p,1-r}(s,a) - \zeta_{p,1-r}(s,a+1)$$
gives definitions of negative integer order zeta functions satisfying
$$\zeta_{p,-r}(s,a) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \langle a + j \rangle^{-s},$$
so that $(-1)^r \zeta_{p,-r}(s,a)$ is the r-th forward difference of the power function $\langle a \rangle^{-s}$ with respect to the a parameter.

Barnes zeta functions of arbitrary integer order

Consider the complex and p-adic multiple zeta functions of order $r \in \mathbb{Z}^+$ defined by

$$\zeta_r(s,a) = \sum_{\bar{t} \in \mathbb{Z}_0^r} (a + |\bar{t}|)^{-s}, \qquad \zeta_{p,r}(s,a) = \frac{1}{(s-1)\cdots(s-r)} \int_{\mathbb{Z}_p^r} \frac{(a+|\bar{t}|)^r}{\langle a + |\bar{t}| \rangle^s} \, d\bar{t},$$

where $|\bar{t}| = t_1 + \cdots + t_r$ denotes the "length" of the vector $\bar{t} = (t_1, \ldots, t_r)$.

- When $r = 0$ these are just the functions $\zeta_0(s,a) = a^{-s}$ and $\zeta_{p,0}(s,a) = \langle a \rangle^{-s}$, and the difference equation

$$\zeta_{p,r}(s,a) - \zeta_{p,r}(s,a+1) = -\zeta_{p,r-1}(s,a)$$

  is satisfied by both $\zeta_r(s,a)$ and $\zeta_{p,r}(s,a)$.
- Rewriting the difference equations

$$\zeta_{p,-r}(s,a) = \zeta_{p,1-r}(s,a) - \zeta_{p,1-r}(s,a+1)$$

  gives definitions of negative integer order zeta functions satisfying

$$\zeta_{p,-r}(s,a) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \langle a + j \rangle^{-s},$$

  so that $(-1)^r \zeta_{p,-r}(s,a)$ is the r-th forward difference of the power function $\langle a \rangle^{-s}$ with respect to the a parameter.
- For any integer r, positive or negative, the a-derivative of $\zeta_{p,r}(s,a)$ is an s-shift

$$\frac{\partial}{\partial a} \zeta_{p,r}(s,a) = -s \frac{\langle a \rangle}{a} \zeta_{p,r}(s+1,a).$$

p-adic Arakawa-Kaneko zeta functions

For $k \in \mathbb{Z}$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ we define the p-adic Arakawa-Kaneko zeta function $\xi_{p,k}(s,a)$ by

$$\xi_{p,k}(s,a) = \sum_{m=0}^{\infty} \frac{\zeta_{p,-m}(s,a)}{(m+1)^k}.$$

This also "defines" the (complex) Arakawa-Kaneko zeta function for $k > 0$, $s \in \mathbb{C}$ and $\Re(a) > 0$.

p-adic Arakawa-Kaneko zeta functions

For $k \in \mathbb{Z}$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ we define the p-adic Arakawa-Kaneko zeta function $\xi_{p,k}(s,a)$ by

$$\xi_{p,k}(s,a) = \sum_{m=0}^{\infty} \frac{\zeta_{p,-m}(s,a)}{(m+1)^k}.$$

This also "defines" the (complex) Arakawa-Kaneko zeta function for $k > 0$, $s \in \mathbb{C}$ and $\Re(a) > 0$.

- For any $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ the function $\xi_{p,k}(s,a)$ is a C^∞ function of s on $\mathbb{Z}_p$ and an analytic function of s on a disc of positive radius about $s = 0$; on this disc it is locally analytic as a function of a and independent of the choice made to define the $\langle \cdot \rangle$ function. If $s \in \mathbb{Z}_p$ the function $\xi_{p,k}(s,a)$ is locally analytic as a function of a on $\mathbb{C}_p \setminus \mathbb{Z}_p$.

p-adic Arakawa-Kaneko zeta functions

For $k \in \mathbb{Z}$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ we define the p-adic Arakawa-Kaneko zeta function $\xi_{p,k}(s,a)$ by

$$\xi_{p,k}(s,a) = \sum_{m=0}^{\infty} \frac{\zeta_{p,-m}(s,a)}{(m+1)^k}.$$

This also "defines" the (complex) Arakawa-Kaneko zeta function for $k > 0$, $s \in \mathbb{C}$ and $\Re(a) > 0$.

- For any $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ the function $\xi_{p,k}(s,a)$ is a C^∞ function of s on $\mathbb{Z}_p$ and an analytic function of s on a disc of positive radius about $s = 0$; on this disc it is locally analytic as a function of a and independent of the choice made to define the $\langle \cdot \rangle$ function. If $s \in \mathbb{Z}_p$ the function $\xi_{p,k}(s,a)$ is locally analytic as a function of a on $\mathbb{C}_p \setminus \mathbb{Z}_p$.

- Arakawa and Kaneko originally defined $\xi_k(s,a)$ as a Mellin transform integral which converges when $\Re(s) > 0$ and $\Re(a) > 0$. The above definition agrees with theirs on that domain and provides an explicit analytic continuation to all $s \in \mathbb{C}$.

p-adic Arakawa-Kaneko zeta functions

For $k \in \mathbb{Z}$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ we define the p-adic Arakawa-Kaneko zeta function $\xi_{p,k}(s,a)$ by

$$\xi_{p,k}(s,a) = \sum_{m=0}^{\infty} \frac{\zeta_{p,-m}(s,a)}{(m+1)^k}.$$

This also "defines" the (complex) Arakawa-Kaneko zeta function for $k > 0$, $s \in \mathbb{C}$ and $\Re(a) > 0$.

- For any $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$ the function $\xi_{p,k}(s,a)$ is a C^∞ function of s on $\mathbb{Z}_p$ and an analytic function of s on a disc of positive radius about $s = 0$; on this disc it is locally analytic as a function of a and independent of the choice made to define the $\langle \cdot \rangle$ function. If $s \in \mathbb{Z}_p$ the function $\xi_{p,k}(s,a)$ is locally analytic as a function of a on $\mathbb{C}_p \setminus \mathbb{Z}_p$.
- Arakawa and Kaneko originally defined $\xi_k(s,a)$ as a Mellin transform integral which converges when $\Re(s) > 0$ and $\Re(a) > 0$. The above definition agrees with theirs on that domain and provides an explicit analytic continuation to all $s \in \mathbb{C}$.
- In the case $k = 1$ the complex version of this definition is consistent with an everywhere-convergent series expansion of the Hurwitz zeta function $\zeta(s,a) = \zeta_1(s,a)$ due to Hasse. This follows from the identity

$$\xi_{p,1}(s,a) = s\frac{\langle a \rangle}{a}\zeta_{p,1}(s+1,a)$$

which holds in $\mathbb{C}_p$ for $|a|_p > 1$ and in $\mathbb{C}$ for $\Re(a) > 0$.

Values at the negative integers

For a positive integer k, the Arakawa-Kaneko zeta function $\xi_k(s,a)$ was defined for $\Re(s) > 0$ and $\Re(a) > 0$ by the Mellin transform integral

$$\Gamma(s)\xi_k(s,a) = \int_0^\infty t^{s-1} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-at}\, dt \qquad \text{where} \qquad \mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

denotes the order k polylogarithm function.

Values at the negative integers

For a positive integer k, the Arakawa-Kaneko zeta function $\xi_k(s,a)$ was defined for $\Re(s) > 0$ and $\Re(a) > 0$ by the Mellin transform integral

$$\Gamma(s)\xi_k(s,a) = \int_0^\infty t^{s-1} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-at}\, dt \qquad \text{where} \qquad \mathrm{Li}_k(z) = \sum_{m=1}^\infty \frac{z^m}{m^k}$$

denotes the order k polylogarithm function.

- The values at the negative integers are the *poly-Bernoulli polynomials* $\mathbb{B}_n^{(k)}(a)$ defined by

$$\xi_k(-n,a) = (-1)^n \mathbb{B}_n^{(k)}(a) \qquad \text{where} \qquad \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-at} = \sum_{n=0}^\infty \mathbb{B}_n^{(k)}(a) \frac{t^n}{n!}.$$

Values at the negative integers

For a positive integer k, the Arakawa-Kaneko zeta function $\xi_k(s,a)$ was defined for $\Re(s) > 0$ and $\Re(a) > 0$ by the Mellin transform integral

$$\Gamma(s)\xi_k(s,a) = \int_0^\infty t^{s-1} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-at}\, dt \qquad \text{where} \qquad \mathrm{Li}_k(z) = \sum_{m=1}^\infty \frac{z^m}{m^k}$$

denotes the order k polylogarithm function.

- The values at the negative integers are the *poly-Bernoulli polynomials* $\mathbb{B}_n^{(k)}(a)$ defined by

$$\xi_k(-n,a) = (-1)^n \mathbb{B}_n^{(k)}(a) \qquad \text{where} \qquad \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-at} = \sum_{n=0}^\infty \mathbb{B}_n^{(k)}(a) \frac{t^n}{n!}.$$

- **Theorem.** For every nonnegative integer n we have

$$\xi_{p,k}(-n,a) = \left(\frac{\langle a \rangle}{a}\right)^n (-1)^n \mathbb{B}_n^{(k)}(a)$$

  for $|a|_p > 1$. Therefore if $a \in \bar{\mathbb{Q}}$ is a complex number with positive real part such that we have $|a|_p > 1$ under our fixed embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$, then

$$\xi_{p,k}(-n,a) = \left(\frac{\langle a \rangle}{a}\right)^n \xi_k(-n,a)$$

  for all nonnegative integers n and k.

Values at the positive integers

Theorem. For every nonnegative integer n we have

$$\xi_{p,k}(n+1,a) = \left(\frac{a}{\langle a\rangle}\right)^{n+1} \sum_{m=0}^{\infty} \frac{m!P_n(h_m^{(1)}(a),...,h_m^{(n)}(a))}{(m+1)^k a(a+1)\cdots(a+m)}$$

for $|a|_p > 1$. Therefore if $a \in \bar{\mathbb{Q}}$ is a complex number with positive real part such that we have $|a|_p > 1$ under our fixed embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$, then

$$\left(\frac{\langle a\rangle}{a}\right)^{n} \xi_{p,k}(n,a) \qquad \text{and} \qquad \xi_k(n,a)$$

are represented by the same convergent series, in $\mathbb{C}_p$ and in $\mathbb{C}$, respectively, for all nonnegative integers n and k.

Theorem. For every nonnegative integer n we have

$$\xi_{p,k}(n+1,a) = \left(\frac{a}{\langle a \rangle}\right)^{n+1} \sum_{m=0}^{\infty} \frac{m! P_n(h_m^{(1)}(a), \ldots, h_m^{(n)}(a))}{(m+1)^k a(a+1)\cdots(a+m)}$$

for $|a|_p > 1$. Therefore if $a \in \bar{\mathbb{Q}}$ is a complex number with positive real part such that we have $|a|_p > 1$ under our fixed embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$, then

$$\left(\frac{\langle a \rangle}{a}\right)^n \xi_{p,k}(n,a) \qquad \text{and} \qquad \xi_k(n,a)$$

are represented by the same convergent series, in $\mathbb{C}_p$ and in $\mathbb{C}$, respectively, for all nonnegative integers n and k.

- Here $P_n(x_1, \ldots, x_n)$ denotes the modified Bell polynomial defined by

$$\exp\left(\sum_{n=1}^{\infty} x_n \frac{t^n}{n}\right) = \sum_{n=0}^{\infty} P_n(x_1, \ldots, x_n) t^n$$

which we evaluate at generalized harmonic numbers

$$h_m^{(n)}(a) = \sum_{j=0}^{m} \frac{1}{(a+j)^n}.$$

Theorem. For every nonnegative integer n we have

$$\xi_{p,k}(n+1,a) = \left(\frac{a}{\langle a\rangle}\right)^{n+1} \sum_{m=0}^{\infty} \frac{m! P_n(h_m^{(1)}(a), \ldots, h_m^{(n)}(a))}{(m+1)^k a(a+1)\cdots(a+m)}$$

for $|a|_p > 1$. Therefore if $a \in \bar{\mathbb{Q}}$ is a complex number with positive real part such that we have $|a|_p > 1$ under our fixed embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$, then

$$\left(\frac{\langle a\rangle}{a}\right)^{n} \xi_{p,k}(n,a) \qquad \text{and} \qquad \xi_k(n,a)$$

are represented by the same convergent series, in $\mathbb{C}_p$ and in $\mathbb{C}$, respectively, for all nonnegative integers n and k.

- Here $P_n(x_1, \ldots, x_n)$ denotes the modified Bell polynomial defined by

$$\exp\left(\sum_{n=1}^{\infty} x_n \frac{t^n}{n}\right) = \sum_{n=0}^{\infty} P_n(x_1, \ldots, x_n) t^n$$

which we evaluate at generalized harmonic numbers

$$h_m^{(n)}(a) = \sum_{j=0}^{m} \frac{1}{(a+j)^n}.$$

- *Proof.* Definition, partial fractions, and derivative-shift property.

Some small examples

The polynomials P_n are determined by their generating function or by the explicit representation

$$P_n(x_1, ..., x_n) = \sum_{k_1+2k_2+3k_3+\cdots=n} \frac{1}{k_1!k_2!k_3!\cdots} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} \left(\frac{x_3}{3}\right)^{k_3} \cdots$$

Some small examples

The polynomials P_n are determined by their generating function or by the explicit representation

$$P_n(x_1, ..., x_n) = \sum_{k_1+2k_2+3k_3+\cdots=n} \frac{1}{k_1!k_2!k_3!\cdots} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} \left(\frac{x_3}{3}\right)^{k_3} \cdots$$

- For $n = 0, 1, 2$ we have

$$\xi_{p,k}(1, a) = \frac{a}{\langle a \rangle} \sum_{m=0}^{\infty} \frac{m!}{(m+1)^k a(a+1)\cdots(a+m)},$$

$$\xi_{p,k}(2, a) = \left(\frac{a}{\langle a \rangle}\right)^2 \sum_{m=0}^{\infty} \frac{m!}{(m+1)^k a(a+1)\cdots(a+m)} \sum_{i=0}^{m} \frac{1}{i+a},$$

$$2\xi_{p,k}(3, a) = \left(\frac{a}{\langle a \rangle}\right)^3 \sum_{m=0}^{\infty} \frac{m!\left(\left(\sum_{i=0}^{m} \frac{1}{i+a}\right)^2 + \sum_{i=0}^{m} \frac{1}{(i+a)^2}\right)}{(m+1)^k a(a+1)\cdots(a+m)},$$

all of which converge p-adically when $|a|_p > 1$ and in $\mathbb{C}$ when $\Re(a) > 0$.

Some small examples

The polynomials P_n are determined by their generating function or by the explicit representation

$$P_n(x_1,\ldots,x_n)=\sum_{k_1+2k_2+3k_3+\cdots=n}\frac{1}{k_1!k_2!k_3!\cdots}\left(\frac{x_1}{1}\right)^{k_1}\left(\frac{x_2}{2}\right)^{k_2}\left(\frac{x_3}{3}\right)^{k_3}\cdots$$

- For $n=0,1,2$ we have

$$\xi_{p,k}(1,a)=\frac{a}{\langle a\rangle}\sum_{m=0}^{\infty}\frac{m!}{(m+1)^k a(a+1)\cdots(a+m)},$$

$$\xi_{p,k}(2,a)=\left(\frac{a}{\langle a\rangle}\right)^2\sum_{m=0}^{\infty}\frac{m!}{(m+1)^k a(a+1)\cdots(a+m)}\sum_{i=0}^{m}\frac{1}{i+a},$$

$$2\xi_{p,k}(3,a)=\left(\frac{a}{\langle a\rangle}\right)^3\sum_{m=0}^{\infty}\frac{m!\left(\left(\sum_{i=0}^m\frac{1}{i+a}\right)^2+\sum_{i=0}^m\frac{1}{(i+a)^2}\right)}{(m+1)^k a(a+1)\cdots(a+m)},$$

  all of which converge p-adically when $|a|_p>1$ and in $\mathbb{C}$ when $\Re(a)>0$.

- In the case $k=0$ we get

$$(a-1)^{-n}=\sum_{m=0}^{\infty}\frac{m!P_{n-1}(h_m^{(1)}(a),\ldots,h_m^{(n-1)}(a))}{a(a+1)\cdots(a+m)}$$

  valid for $\Re(a)>1$ and for $|a|_p>1$.

A challenging example

Ramanujan obtained the explicit evaluation

$$\xi_2(1, \frac{1}{2}) := \sum_{m=1}^{\infty} \frac{4^m}{m^3 \binom{2m}{m}} = \pi^2 \log 2 - \frac{7}{2}\zeta(3) \quad \text{in} \quad \mathbb{R}$$

by connecting the integral representation definition to log-sine integrals.

A challenging example

Ramanujan obtained the explicit evaluation

$$\xi_2(1, \frac{1}{2}) := \sum_{m=1}^{\infty} \frac{4^m}{m^3 \binom{2m}{m}} = \pi^2 \log 2 - \frac{7}{2}\zeta(3) \quad \text{in} \quad \mathbb{R}$$

by connecting the integral representation definition to log-sine integrals.

- I recently proved the analogous 2-adic identity

$$2\xi_{2,2}(1, \frac{1}{2}) := \sum_{m=1}^{\infty} \frac{4^m}{m^3 \binom{2m}{m}} = -4\zeta_{2,1}(3, \frac{1}{2}) \quad \text{in} \quad \mathbb{Q}_2$$

  in a rather *ad hoc* fashion.

A challenging example

Ramanujan obtained the explicit evaluation

$$\xi_2(1, \frac{1}{2}) := \sum_{m=1}^{\infty} \frac{4^m}{m^3 \binom{2m}{m}} = \pi^2 \log 2 - \frac{7}{2}\zeta(3) \quad \text{in} \quad \mathbb{R}$$

by connecting the integral representation definition to log-sine integrals.

- I recently proved the analogous 2-adic identity

$$2\xi_{2,2}(1, \frac{1}{2}) := \sum_{m=1}^{\infty} \frac{4^m}{m^3 \binom{2m}{m}} = -4\zeta_{2,1}(3, \frac{1}{2}) \quad \text{in} \quad \mathbb{Q}_2$$

  in a rather *ad hoc* fashion.

- This may be regarded as a reasonable 2-adic analogue because

$$\log_2 2 = 0 \qquad \text{and} \qquad \zeta(3, \frac{1}{2}) = 7\zeta(3) \qquad \text{and} \qquad \left(\frac{\langle 1/2 \rangle}{1/2}\right)^3 = 8.$$

A challenging example

Ramanujan obtained the explicit evaluation

$$\xi_2(1, \frac{1}{2}) := \sum_{m=1}^{\infty} \frac{4^m}{m^3 \binom{2m}{m}} = \pi^2 \log 2 - \frac{7}{2}\zeta(3) \quad \text{in} \quad \mathbb{R}$$

by connecting the integral representation definition to log-sine integrals.

- I recently proved the analogous 2-adic identity

$$2\xi_{2,2}(1, \frac{1}{2}) := \sum_{m=1}^{\infty} \frac{4^m}{m^3 \binom{2m}{m}} = -4\zeta_{2,1}(3, \frac{1}{2}) \quad \text{in} \quad \mathbb{Q}_2$$

  in a rather *ad hoc* fashion.
- This may be regarded as a reasonable 2-adic analogue because

$$\log_2 2 = 0 \qquad \text{and} \qquad \zeta(3, \frac{1}{2}) = 7\zeta(3) \qquad \text{and} \qquad \left(\frac{\langle 1/2 \rangle}{1/2}\right)^3 = 8.$$

- Interpreting π p-adically is tricky, but there are good reasons why π should be zero. For example if α is any root of unity then $\log \alpha \in 2\pi i\mathbb{Q}$ but $\log_p \alpha = 0$.

A challenging example

Ramanujan obtained the explicit evaluation

$$\xi_2(1, \frac{1}{2}) := \sum_{m=1}^{\infty} \frac{4^m}{m^3 \binom{2m}{m}} = \pi^2 \log 2 - \frac{7}{2}\zeta(3) \quad \text{in} \quad \mathbb{R}$$

by connecting the integral representation definition to log-sine integrals.

- I recently proved the analogous 2-adic identity

$$2\xi_{2,2}(1, \frac{1}{2}) := \sum_{m=1}^{\infty} \frac{4^m}{m^3 \binom{2m}{m}} = -4\zeta_{2,1}(3, \frac{1}{2}) \quad \text{in} \quad \mathbb{Q}_2$$

  in a rather *ad hoc* fashion.
- This may be regarded as a reasonable 2-adic analogue because

$$\log_2 2 = 0 \qquad \text{and} \qquad \zeta(3, \frac{1}{2}) = 7\zeta(3) \qquad \text{and} \qquad \left(\frac{\langle 1/2 \rangle}{1/2}\right)^3 = 8.$$

- Interpreting π p-adically is tricky, but there are good reasons why π should be zero. For example if α is any root of unity then $\log \alpha \in 2\pi i\mathbb{Q}$ but $\log_p \alpha = 0$.
- They also gave similar evaluations for $\xi_2(2, \frac{1}{2})$ and $\xi_3(1, \frac{1}{2})$ in terms of π, logs, zeta values, and the Ramanujan constant $C = \sum_{n=1}^{\infty}(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1})/(2n)^3 = 0.1622719\cdots$. It should be very interesting to understand the 2-adic analogues of these as well.

Limit formulas

Corollary. For any positive integer s and any sequence of positive integers n which converge p-adically to $-s$ (e.g., $n_r = p^r - s$ as $r \to \infty$) we have the limit formula

$$\lim_{n \to -s} (-1)^n \left(\frac{\langle a \rangle}{a} \right)^{n+s} \mathbb{B}_n^{(k)}(a) = \sum_{m=0}^{\infty} \frac{m! P_{s-1}(h_m^{(1)}(a), \ldots, h_m^{(s-1)}(a))}{(m+1)^k a(a+1) \cdots (a+m)}$$

in $\mathbb{C}_p$, valid when $|a|_p > 1$.

Corollary. For any positive integer s and any sequence of positive integers n which converge p-adically to $-s$ (e.g., $n_r = p^r - s$ as $r \to \infty$) we have the limit formula

$$\lim_{n \to -s} (-1)^n \left(\frac{\langle a \rangle}{a} \right)^{n+s} \mathbb{B}_n^{(k)}(a) = \sum_{m=0}^{\infty} \frac{m! P_{s-1}(h_m^{(1)}(a), \ldots, h_m^{(s-1)}(a))}{(m+1)^k a(a+1)\cdots(a+m)}$$

in $\mathbb{C}_p$, valid when $|a|_p > 1$.

- Conversely for any positive integer s and any sequence of positive integers n which converge p-adically to $-s$ we have

$$\lim_{n \to -s} (-1)^{s-1} \left(\frac{a}{\langle a \rangle} \right)^{n+s} \sum_{m=0}^{\infty} \frac{m! P_n(h_m^{(1)}(a), \ldots, h_m^{(n)}(a))}{(m+1)^k a(a+1)\cdots(a+m)} = \mathbb{B}_{s-1}^{(k)}(a)$$

  in $\mathbb{C}_p$, valid when $|a|_p > 1$.

Limit formulas

Corollary. For any positive integer s and any sequence of positive integers n which converge p-adically to $-s$ (e.g., $n_r = p^r - s$ as $r \to \infty$) we have the limit formula

$$\lim_{n\to -s} (-1)^n \left(\frac{\langle a\rangle}{a}\right)^{n+s} \mathbb{B}_n^{(k)}(a) = \sum_{m=0}^{\infty} \frac{m! P_{s-1}(h_m^{(1)}(a), \ldots, h_m^{(s-1)}(a))}{(m+1)^k a(a+1)\cdots(a+m)}$$

in $\mathbb{C}_p$, valid when $|a|_p > 1$.

- Conversely for any positive integer s and any sequence of positive integers n which converge p-adically to $-s$ we have

$$\lim_{n\to -s} (-1)^{s-1} \left(\frac{a}{\langle a\rangle}\right)^{n+s} \sum_{m=0}^{\infty} \frac{m! P_n(h_m^{(1)}(a), \ldots, h_m^{(n)}(a))}{(m+1)^k a(a+1)\cdots(a+m)} = \mathbb{B}_{s-1}^{(k)}(a)$$

  in $\mathbb{C}_p$, valid when $|a|_p > 1$.
- *Proof.* These equations merely express the fact that $\xi_{p,k}(s, a)$ is a continuous function of s on $\mathbb{Z}_p$.

Limit formulas

Corollary. For any positive integer s and any sequence of positive integers n which converge p-adically to $-s$ (e.g., $n_r = p^r - s$ as $r \to \infty$) we have the limit formula

$$\lim_{n \to -s} (-1)^n \left(\frac{\langle a \rangle}{a} \right)^{n+s} \mathbb{B}_n^{(k)}(a) = \sum_{m=0}^{\infty} \frac{m! P_{s-1}(h_m^{(1)}(a), \ldots, h_m^{(s-1)}(a))}{(m+1)^k a(a+1) \cdots (a+m)}$$

in $\mathbb{C}_p$, valid when $|a|_p > 1$.

- Conversely for any positive integer s and any sequence of positive integers n which converge p-adically to $-s$ we have

  $$\lim_{n \to -s} (-1)^{s-1} \left(\frac{a}{\langle a \rangle} \right)^{n+s} \sum_{m=0}^{\infty} \frac{m! P_n(h_m^{(1)}(a), \ldots, h_m^{(n)}(a))}{(m+1)^k a(a+1) \cdots (a+m)} = \mathbb{B}_{s-1}^{(k)}(a)$$

  in $\mathbb{C}_p$, valid when $|a|_p > 1$.
- *Proof.* These equations merely express the fact that $\xi_{p,k}(s, a)$ is a continuous function of s on $\mathbb{Z}_p$.
- Of course, these formulas are p-adic only, they are not real nor complex.

Corollary. For any positive integer s and any sequence of positive integers n which converge p-adically to $-s$ (e.g., $n_r = p^r - s$ as $r \to \infty$) we have the limit formula

$$\lim_{n\to -s}(-1)^n\left(\frac{\langle a\rangle}{a}\right)^{n+s}\mathbb{B}_n^{(k)}(a)=\sum_{m=0}^{\infty}\frac{m!P_{s-1}(h_m^{(1)}(a),\ldots,h_m^{(s-1)}(a))}{(m+1)^k a(a+1)\cdots(a+m)}$$

in $\mathbb{C}_p$, valid when $|a|_p > 1$.

- Conversely for any positive integer s and any sequence of positive integers n which converge p-adically to $-s$ we have

$$\lim_{n\to -s}(-1)^{s-1}\left(\frac{a}{\langle a\rangle}\right)^{n+s}\sum_{m=0}^{\infty}\frac{m!P_n(h_m^{(1)}(a),\ldots,h_m^{(n)}(a))}{(m+1)^k a(a+1)\cdots(a+m)}=\mathbb{B}_{s-1}^{(k)}(a)$$

  in $\mathbb{C}_p$, valid when $|a|_p > 1$.

- *Proof.* These equations merely express the fact that $\xi_{p,k}(s,a)$ is a continuous function of s on $\mathbb{Z}_p$.
- Of course, these formulas are p-adic only, they are not real nor complex.
- The previous “challenging” 2-adic identity is equivalent to showing

$$2^{2^r-2}\left(\mathbb{B}_{2^r-2}^{(1)}\left(\frac{1}{2}\right)-4\mathbb{B}_{2^r-1}^{(2)}\left(\frac{1}{2}\right)\right)\to 0 \quad \text{in} \quad \mathbb{Q}_2.$$

Hurwitz zeta function examples

Taking $k = 1$, $p = 2$, $a = 1/2$, and $n = 1, 2, 3$ in this theorem yields

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2 \binom{2m}{m}} = \begin{cases} 3\zeta(2) = \frac{\pi^2}{2} & \text{in} \quad \mathbb{R} \\ 4\zeta_{2,1}(2, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2 \binom{2m}{m}} \left(H_{2m-1} - \frac{H_{m-1}}{2} \right) = \begin{cases} 7\zeta(3) & \text{in} \quad \mathbb{R} \\ 8\zeta_{2,1}(3, \frac{1}{2}) & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2 \binom{2m}{m}} \left(\left(H_{2m-1} - \frac{H_{m-1}}{2} \right)^2 + \left(H_{2m-1}^{(2)} - \frac{H_{m-1}^{(2)}}{4} \right) \right) = \begin{cases} \frac{45}{2}\zeta(4) = \frac{\pi^4}{4} & \text{in} \quad \mathbb{R} \\ 24\zeta_{2,1}(4, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2, \end{cases}$$

where $H_m^{(n)} := \sum_{j=1}^{m} 1/j^n$ denotes the generalized harmonic number, with $H_m := H_m^{(1)}$.

Hurwitz zeta function examples

Taking $k = 1$, $p = 2$, $a = 1/2$, and $n = 1, 2, 3$ in this theorem yields

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2\binom{2m}{m}} = \begin{cases} 3\zeta(2) = \frac{\pi^2}{2} & \text{in} \quad \mathbb{R} \\ 4\zeta_{2,1}(2, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2\binom{2m}{m}} \left(H_{2m-1} - \frac{H_{m-1}}{2} \right) = \begin{cases} 7\zeta(3) & \text{in} \quad \mathbb{R} \\ 8\zeta_{2,1}(3, \frac{1}{2}) & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2\binom{2m}{m}} \left(\left(H_{2m-1} - \frac{H_{m-1}}{2} \right)^2 + \left(H_{2m-1}^{(2)} - \frac{H_{m-1}^{(2)}}{4} \right) \right) = \begin{cases} \frac{45}{2}\zeta(4) = \frac{\pi^4}{4} & \text{in} \quad \mathbb{R} \\ 24\zeta_{2,1}(4, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2, \end{cases}$$

where $H_m^{(n)} := \sum_{j=1}^{m} 1/j^n$ denotes the generalized harmonic number, with $H_m := H_m^{(1)}$.

- When n is a positive even integer we have $\zeta_{2,1}(n, \frac{1}{2}) = 0$ by the reflection formula

$$\zeta_{p,r}(s, a) = (-1)^r \langle -1 \rangle^{-s} \zeta_{p,r}(s, r - a).$$

Hurwitz zeta function examples

Taking $k=1$, $p=2$, $a=1/2$, and $n=1,2,3$ in this theorem yields

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2\binom{2m}{m}} = \begin{cases} 3\zeta(2) = \frac{\pi^2}{2} & \text{in} \quad \mathbb{R} \\ 4\zeta_{2,1}(2, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2\binom{2m}{m}} \left(H_{2m-1} - \frac{H_{m-1}}{2} \right) = \begin{cases} 7\zeta(3) & \text{in} \quad \mathbb{R} \\ 8\zeta_{2,1}(3, \frac{1}{2}) & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2\binom{2m}{m}} \left(\left(H_{2m-1} - \frac{H_{m-1}}{2} \right)^2 + \left(H_{2m-1}^{(2)} - \frac{H_{m-1}^{(2)}}{4} \right) \right) = \begin{cases} \frac{45}{2}\zeta(4) = \frac{\pi^4}{4} & \text{in} \quad \mathbb{R} \\ 24\zeta_{2,1}(4, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2, \end{cases}$$

where $H_m^{(n)} := \sum_{j=1}^{m} 1/j^n$ denotes the generalized harmonic number, with $H_m := H_m^{(1)}$.

- When n is a positive even integer we have $\zeta_{2,1}(n, \frac{1}{2}) = 0$ by the reflection formula

$$\zeta_{p,r}(s,a) = (-1)^r \langle -1 \rangle^{-s} \zeta_{p,r}(s, r-a).$$

- When n is an odd positive integer the value $\zeta_{2,1}(n, \frac{1}{2}) = L_2(n, \chi_0)$ is a value of the 2-adic Kubota-Leopoldt L-function for the trivial Dirichlet character χ_0, that is, a 2-adic zeta value.

Taking $k = 1$, $p = 2$, $a = 1/2$, and $n = 1, 2, 3$ in this theorem yields

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2 \binom{2m}{m}} = \begin{cases} 3\zeta(2) = \frac{\pi^2}{2} & \text{in} \quad \mathbb{R} \\ 4\zeta_{2,1}(2, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2 \binom{2m}{m}} \left(H_{2m-1} - \frac{H_{m-1}}{2} \right) = \begin{cases} 7\zeta(3) & \text{in} \quad \mathbb{R} \\ 8\zeta_{2,1}(3, \frac{1}{2}) & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2 \binom{2m}{m}} \left(\left(H_{2m-1} - \frac{H_{m-1}}{2} \right)^2 + \left(H_{2m-1}^{(2)} - \frac{H_{m-1}^{(2)}}{4} \right) \right) = \begin{cases} \frac{45}{2}\zeta(4) = \frac{\pi^4}{4} & \text{in} \quad \mathbb{R} \\ 24\zeta_{2,1}(4, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2, \end{cases}$$

where $H_m^{(n)} := \sum_{j=1}^{m} 1/j^n$ denotes the generalized harmonic number, with $H_m := H_m^{(1)}$.

- When n is a positive even integer we have $\zeta_{2,1}(n, \frac{1}{2}) = 0$ by the reflection formula

$$\zeta_{p,r}(s, a) = (-1)^r \langle -1 \rangle^{-s} \zeta_{p,r}(s, r - a).$$

- When n is an odd positive integer the value $\zeta_{2,1}(n, \frac{1}{2}) = L_2(n, \chi_0)$ is a value of the 2-adic Kubota-Leopoldt L-function for the trivial Dirichlet character χ_0, that is, a 2-adic zeta value.
- The complex zeta values are related by an identity $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$.

Hurwitz zeta function examples

Taking $k = 1$, $p = 2$, $a = 1/2$, and $n = 1, 2, 3$ in this theorem yields

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2 \binom{2m}{m}} = \begin{cases} 3\zeta(2) = \frac{\pi^2}{2} & \text{in} \quad \mathbb{R} \\ 4\zeta_{2,1}(2, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2 \binom{2m}{m}} \left(H_{2m-1} - \frac{H_{m-1}}{2} \right) = \begin{cases} 7\zeta(3) & \text{in} \quad \mathbb{R} \\ 8\zeta_{2,1}(3, \frac{1}{2}) & \text{in} \quad \mathbb{Q}_2; \end{cases}$$

$$\sum_{m=1}^{\infty} \frac{4^m}{m^2 \binom{2m}{m}} \left(\left(H_{2m-1} - \frac{H_{m-1}}{2} \right)^2 + \left(H_{2m-1}^{(2)} - \frac{H_{m-1}^{(2)}}{4} \right) \right) = \begin{cases} \frac{45}{2}\zeta(4) = \frac{\pi^4}{4} & \text{in} \quad \mathbb{R} \\ 24\zeta_{2,1}(4, \frac{1}{2}) = 0 & \text{in} \quad \mathbb{Q}_2, \end{cases}$$

where $H_m^{(n)} := \sum_{j=1}^{m} 1/j^n$ denotes the generalized harmonic number, with $H_m := H_m^{(1)}$.

- When n is a positive even integer we have $\zeta_{2,1}(n, \frac{1}{2}) = 0$ by the reflection formula

$$\zeta_{p,r}(s, a) = (-1)^r \langle -1 \rangle^{-s} \zeta_{p,r}(s, r - a).$$

- When n is an odd positive integer the value $\zeta_{2,1}(n, \frac{1}{2}) = L_2(n, \chi_0)$ is a value of the 2-adic Kubota-Leopoldt L-function for the trivial Dirichlet character χ_0, that is, a 2-adic zeta value.
- The complex zeta values are related by an identity $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$.
- The irrationality of $\zeta(3)$ was proved by Apéry in 1978, and the irrationality of $\zeta_{2,1}(3, \frac{1}{2})$ was proved by Calegari in 2004.

A p-adic Arakawa-Kaneko digamma function

We may define a p-adic Arakawa-Kaneko digamma function

$$\psi_{p,k}(a) := -\frac{\partial}{\partial s}\xi_{p,k}(s,a)\bigg|_{s=0}$$

for $k \in \mathbb{Z}^+$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$, or $\Re(a) > 0$.

A p-adic Arakawa-Kaneko digamma function

We may define a p-adic Arakawa-Kaneko digamma function

$$\psi_{p,k}(a) := -\frac{\partial}{\partial s}\xi_{p,k}(s,a)\bigg|_{s=0}$$

for $k \in \mathbb{Z}^+$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$, or $\Re(a) > 0$.

- We have $\psi_{p,0}(a) = \log_p(a-1)$ for $|a|_p > 1$, and $\psi_{p,1}$ is equal to the p-adic digamma function $\psi_p(a)$, which is defined as the derivative of the p-adic log gamma function.

A p-adic Arakawa-Kaneko digamma function

We may define a p-adic Arakawa-Kaneko digamma function

$$\psi_{p,k}(a) := -\frac{\partial}{\partial s}\xi_{p,k}(s,a)\Big|_{s=0}$$

for $k \in \mathbb{Z}^+$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$, or $\Re(a) > 0$.

- We have $\psi_{p,0}(a) = \log_p(a-1)$ for $|a|_p > 1$, and $\psi_{p,1}$ is equal to the p-adic digamma function $\psi_p(a)$, which is defined as the derivative of the p-adic log gamma function.
- The value $-\psi_{p,1}(a)$ is equal to $\langle a \rangle / a$ times the constant term in the Laurent expansion of $\zeta_{p,1}(s,a)$ at $s = 1$, which in $\mathbb{C}$ is the negative of Euler's constant $\gamma = -\psi(1)$.

A p-adic Arakawa-Kaneko digamma function

We may define a p-adic Arakawa-Kaneko digamma function

$$\psi_{p,k}(a) := -\frac{\partial}{\partial s}\xi_{p,k}(s,a)\bigg|_{s=0}$$

for $k \in \mathbb{Z}^+$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$, or $\Re(a) > 0$.

- We have $\psi_{p,0}(a) = \log_p(a-1)$ for $|a|_p > 1$, and $\psi_{p,1}$ is equal to the p-adic digamma function $\psi_p(a)$, which is defined as the derivative of the p-adic log gamma function.
- The value $-\psi_{p,1}(a)$ is equal to $\langle a \rangle / a$ times the constant term in the Laurent expansion of $\zeta_{p,1}(s,a)$ at $s=1$, which in $\mathbb{C}$ is the negative of Euler's constant $\gamma = -\psi(1)$.
- For $|a|_p > 1$ there is a limit formula in terms of $\mathbb{B}_n^{(k)}$, and from the theorem we get the expansion

$$\psi_{p,k}(a) = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\frac{\log_p(a+j)}{(m+1)^k}.$$

A p-adic Arakawa-Kaneko digamma function

We may define a p-adic Arakawa-Kaneko digamma function

$$\psi_{p,k}(a) := -\frac{\partial}{\partial s}\xi_{p,k}(s,a)\bigg|_{s=0}$$

for $k \in \mathbb{Z}^+$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$, or $\Re(a) > 0$.

- We have $\psi_{p,0}(a) = \log_p(a-1)$ for $|a|_p > 1$, and $\psi_{p,1}$ is equal to the p-adic digamma function $\psi_p(a)$, which is defined as the derivative of the p-adic log gamma function.
- The value $-\psi_{p,1}(a)$ is equal to $\langle a \rangle / a$ times the constant term in the Laurent expansion of $\zeta_{p,1}(s,a)$ at $s=1$, which in $\mathbb{C}$ is the negative of Euler's constant $\gamma = -\psi(1)$.
- For $|a|_p > 1$ there is a limit formula in terms of $\mathbb{B}_n^{(k)}$, and from the theorem we get the expansion

$$\psi_{p,k}(a) = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\frac{\log_p(a+j)}{(m+1)^k}.$$

- As an example, computing $\psi_{2,1}(3/4) - \psi_{2,1}(1/4)$ gives

$$\sum_{m=0}^{\infty}\frac{1}{m+1}\sum_{j=0}^{m}\binom{m}{j}(-1)^j\log_2\left(\frac{4j+3}{4j+1}\right) = \begin{cases}\pi & \text{in} \quad \mathbb{R},\\ 0 & \text{in} \quad \mathbb{Q}_2.\end{cases}$$

A p-adic Lerch transcendent

The *Lerch transcendent* $\Phi(z,s,a)$ is defined for $\Re(a) > 0$ by the series

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

which converges when either $|z| < 1$, or $|z| = 1$ and $\Re(s) > 1$; it may be analytically continued to all $s \in \mathbb{C}$ when $z \in \mathbb{C} \setminus [1, +\infty)$, and to $s \in \mathbb{C} \setminus \{1\}$ when $z = 1$.

A p-adic Lerch transcendent

The *Lerch transcendent* $\Phi(z, s, a)$ is defined for $\Re(a) > 0$ by the series

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

which converges when either $|z| < 1$, or $|z| = 1$ and $\Re(s) > 1$; it may be analytically continued to all $s \in \mathbb{C}$ when $z \in \mathbb{C} \setminus [1, +\infty)$, and to $s \in \mathbb{C} \setminus \{1\}$ when $z = 1$.

- We define the p-adic analogue $\Phi_p(z, s, a)$ by the series

$$(1-z)\Phi_p(z, s, a) = \sum_{m=0}^{\infty} \left(\frac{-z}{1-z} \right)^m \zeta_{p,-m}(s, a),$$

  since this complex series converges when $\Re(z) < 1/2$ and agrees there with the definition of $\Phi(z, s, a)$.

The *Lerch transcendent* $\Phi(z,s,a)$ is defined for $\Re(a) > 0$ by the series

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

which converges when either $|z| < 1$, or $|z| = 1$ and $\Re(s) > 1$; it may be analytically continued to all $s \in \mathbb{C}$ when $z \in \mathbb{C} \setminus [1, +\infty)$, and to $s \in \mathbb{C} \setminus \{1\}$ when $z = 1$.

- We define the p-adic analogue $\Phi_p(z,s,a)$ by the series

$$(1-z)\Phi_p(z,s,a) = \sum_{m=0}^{\infty} \left(\frac{-z}{1-z}\right)^m \zeta_{p,-m}(s,a),$$

  since this complex series converges when $\Re(z) < 1/2$ and agrees there with the definition of $\Phi(z,s,a)$.

- **Theorem.** For $|z-1|_p \geq 1$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$, the function $\Phi_p(z,s,a)$ is a C^∞ function of s on $\mathbb{Z}_p$ and an analytic function of s on a disc of positive radius about $s=0$. If $|z-1|_p < 1$, $|a|_p > 1$, and $|(z-1)a|_p > p^{-1/(p-1)}$, the function $\Phi_p(z,s,a)$ is a C^∞ function of s on $\mathbb{Z}_p$.

A p-adic Lerch transcendent

The *Lerch transcendent* $\Phi(z,s,a)$ is defined for $\Re(a) > 0$ by the series

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

which converges when either $|z| < 1$, or $|z| = 1$ and $\Re(s) > 1$; it may be analytically continued to all $s \in \mathbb{C}$ when $z \in \mathbb{C} \setminus [1, +\infty)$, and to $s \in \mathbb{C} \setminus \{1\}$ when $z = 1$.

- We define the p-adic analogue $\Phi_p(z,s,a)$ by the series

$$(1-z)\Phi_p(z,s,a) = \sum_{m=0}^{\infty} \left(\frac{-z}{1-z}\right)^m \zeta_{p,-m}(s,a),$$

  since this complex series converges when $\Re(z) < 1/2$ and agrees there with the definition of $\Phi(z,s,a)$.

- **Theorem.** For $|z-1|_p \geq 1$ and $a \in \mathbb{C}_p \setminus \mathbb{Z}_p$, the function $\Phi_p(z,s,a)$ is a C^∞ function of s on $\mathbb{Z}_p$ and an analytic function of s on a disc of positive radius about $s=0$. If $|z-1|_p < 1$, $|a|_p > 1$, and $|(z-1)a|_p > p^{-1/(p-1)}$, the function $\Phi_p(z,s,a)$ is a C^∞ function of s on $\mathbb{Z}_p$.

- *Proof.* Under these conditions the above series converges uniformly, by previous results and estimates that aren't that hard.

Values at the integers

It is apparent that $\Phi(z, -m, a)$ is a polynomial of degree m in a when m is a nonnegative integer, and

$$\Phi_p(z, -m, a) = \left(\frac{\langle a \rangle}{a}\right)^m \Phi(z, -m, a)$$

when $|a|_p > 1$. In the special case $z = -1$ the values $\Phi(-1, -m, a) = \frac{1}{2} E_m(a)$ are called *Euler polynomials*.

Values at the integers

It is apparent that $\Phi(z,-m,a)$ is a polynomial of degree m in a when m is a nonnegative integer, and

$$\Phi_p(z,-m,a) = \left(\frac{\langle a\rangle}{a}\right)^m \Phi(z,-m,a)$$

when $|a|_p > 1$. In the special case $z=-1$ the values $\Phi(-1,-m,a) = \frac{1}{2}E_m(a)$ are called *Euler polynomials*.

- **Theorem.** For every nonnegative integer n we have

$$(1-z)\Phi_p(z,n+1,a) = \left(\frac{a}{\langle a\rangle}\right)^{n+1} \sum_{m=0}^{\infty} \left(\frac{-z}{1-z}\right)^m \frac{m!P_n(h_m^{(1)}(a),\ldots,h_m^{(n)}(a))}{a(a+1)\cdots(a+m)}$$

when $|a|_p > 1$ and $|(z-1)a|_p > p^{-1/(p-1)}$. Therefore if $a,z \in \bar{\mathbb{Q}}$ are complex numbers with $\Re(a) > 0$ and $\Re(z) < 1/2$ such that we have $|a|_p > 1$ and $|(z-1)a|_p > p^{-1/(p-1)}$ under our fixed embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$, then

$$\left(\frac{\langle a\rangle}{a}\right)^n \Phi_p(z,n,a) \qquad \text{and} \qquad \Phi(z,n,a)$$

are represented by the same convergent series, in $\mathbb{C}_p$ and in $\mathbb{C}$, respectively, for all positive integers n.

Values at the integers

It is apparent that $\Phi(z,-m,a)$ is a polynomial of degree m in a when m is a nonnegative integer, and

$$\Phi_p(z,-m,a) = \left(\frac{\langle a\rangle}{a}\right)^m \Phi(z,-m,a)$$

when $|a|_p > 1$. In the special case $z=-1$ the values $\Phi(-1,-m,a) = \frac{1}{2}E_m(a)$ are called *Euler polynomials*.

- **Theorem.** For every nonnegative integer n we have

$$(1-z)\Phi_p(z,n+1,a) = \left(\frac{a}{\langle a\rangle}\right)^{n+1} \sum_{m=0}^{\infty} \left(\frac{-z}{1-z}\right)^m \frac{m! P_n(h_m^{(1)}(a),\ldots,h_m^{(n)}(a))}{a(a+1)\cdots(a+m)}$$

  when $|a|_p > 1$ and $|(z-1)a|_p > p^{-1/(p-1)}$. Therefore if $a,z \in \bar{\mathbb{Q}}$ are complex numbers with $\Re(a) > 0$ and $\Re(z) < 1/2$ such that we have $|a|_p > 1$ and $|(z-1)a|_p > p^{-1/(p-1)}$ under our fixed embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}_p$, then

$$\left(\frac{\langle a\rangle}{a}\right)^n \Phi_p(z,n,a) \qquad \text{and} \qquad \Phi(z,n,a)$$

  are represented by the same convergent series, in $\mathbb{C}_p$ and in $\mathbb{C}$, respectively, for all positive integers n.

- *Proof.* Definition, partial fractions, and derivative shift identity.

Special cases when $z = -1$

For odd primes p Kim and Hu defined a p-adic Euler zeta function $\zeta_{p,E}(s,x)$ which p-adically interpolates the Euler polynomials at negative integer values of s; by comparison of the respective values we have

$$\zeta_{p,E}(s,x) = 2\Phi_p(-1, s+1, x)$$

when $|x|_p > 1$ and p is an odd prime.

Special cases when $z = -1$

For odd primes p Kim and Hu defined a p-adic Euler zeta function $\zeta_{p,E}(s,x)$ which p-adically interpolates the Euler polynomials at negative integer values of s; by comparison of the respective values we have

$$\zeta_{p,E}(s,x) = 2\Phi_p(-1, s+1, x)$$

when $|x|_p > 1$ and p is an odd prime.

- Thus we obtain expansions

$$\zeta_{p,E}(n,a) = \left(\frac{a}{\langle a\rangle}\right)^{n+1} \sum_{m=0}^{\infty} \frac{m! P_n(h_m^{(1)}(a), \ldots, h_m^{(n)}(a))}{2^m a(a+1)\cdots(a+m)}$$

  of positive-integer values of $\zeta_{p,E}(s,a)$ in terms of generalized harmonic numbers, valid when $|a|_p > 1$ for odd primes p.

Special cases when $z = -1$

For odd primes p Kim and Hu defined a p-adic Euler zeta function $\zeta_{p,E}(s,x)$ which p-adically interpolates the Euler polynomials at negative integer values of s; by comparison of the respective values we have

$$\zeta_{p,E}(s,x) = 2\Phi_p(-1, s+1, x)$$

when $|x|_p > 1$ and p is an odd prime.

- Thus we obtain expansions

$$\zeta_{p,E}(n,a) = \left(\frac{a}{\langle a\rangle}\right)^{n+1} \sum_{m=0}^{\infty} \frac{m! P_n(h_m^{(1)}(a), \dots, h_m^{(n)}(a))}{2^m a(a+1)\cdots(a+m)}$$

  of positive-integer values of $\zeta_{p,E}(s,a)$ in terms of generalized harmonic numbers, valid when $|a|_p > 1$ for odd primes p.

- When $p = 2$ and $a = 1/2$ we obtain the expansions

$$2^{-(n+2)} \sum_{m=0}^{\infty} \frac{2^{m+1} P_n(h_m^{(1)}(1/2), \dots, h_m^{(n)}(1/2))}{(2m+1)\binom{2m}{m}} = \begin{cases} \beta(n+1) & \text{in} \quad \mathbb{R} \\ \beta_2(n+1) & \text{in} \quad \mathbb{Q}_2 \end{cases}$$

  for the complex and 2-adic *Dirichlet beta functions*, defined by

$$\beta(s) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 2^{-s}\Phi(-1, s, \frac{1}{2}), \qquad \beta_2(s) := \Phi_2(-1, s, \frac{1}{2}).$$

Beta value examples

For $n = 0, 1, 2$ our expansion yields

$$\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)\binom{2m}{m}} = \begin{cases} \beta(1) = \pi/4 & \text{in} \quad \mathbb{R} \\ \beta_2(1) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

$$\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)\binom{2m}{m}} \left(H_{2m+1} - \frac{H_m}{2} \right) = \begin{cases} \beta(2) = G & \text{in} \quad \mathbb{R} \\ \beta_2(2) = G_2 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

$$\sum_{m=0}^{\infty} \frac{2^{m-2}}{(2m+1)\binom{2m}{m}} \left(\left(H_{2m+1} - \frac{H_m}{2} \right)^2 + \left(H_{2m+1}^{(2)} - \frac{H_m^{(2)}}{4} \right) \right) = \begin{cases} \beta(3) = \frac{\pi^3}{32} & \text{in} \quad \mathbb{R} \\ \beta_2(3) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

Beta value examples

For $n = 0, 1, 2$ our expansion yields

$$\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)\binom{2m}{m}} = \begin{cases} \beta(1) = \pi/4 & \text{in} \quad \mathbb{R} \\ \beta_2(1) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

$$\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)\binom{2m}{m}} \left(H_{2m+1} - \frac{H_m}{2} \right) = \begin{cases} \beta(2) = G & \text{in} \quad \mathbb{R} \\ \beta_2(2) = G_2 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

$$\sum_{m=0}^{\infty} \frac{2^{m-2}}{(2m+1)\binom{2m}{m}} \left(\left(H_{2m+1} - \frac{H_m}{2} \right)^2 + \left(H_{2m+1}^{(2)} - \frac{H_m^{(2)}}{4} \right) \right) = \begin{cases} \beta(3) = \frac{\pi^3}{32} & \text{in} \quad \mathbb{R} \\ \beta_2(3) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

- Since the classical Euler number $E_m = 2^m E_m(1/2) = 0$ when m is odd, the 2-adic value $\beta_2(m) = 0$ for every odd integer m.

Beta value examples

For $n = 0, 1, 2$ our expansion yields

$$\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)\binom{2m}{m}} = \begin{cases} \beta(1) = \pi/4 & \text{in} \quad \mathbb{R} \\ \beta_2(1) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

$$\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)\binom{2m}{m}} \left(H_{2m+1} - \frac{H_m}{2} \right) = \begin{cases} \beta(2) = G & \text{in} \quad \mathbb{R} \\ \beta_2(2) = G_2 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

$$\sum_{m=0}^{\infty} \frac{2^{m-2}}{(2m+1)\binom{2m}{m}} \left(\left(H_{2m+1} - \frac{H_m}{2} \right)^2 + \left(H_{2m+1}^{(2)} - \frac{H_m^{(2)}}{4} \right) \right) = \begin{cases} \beta(3) = \frac{\pi^3}{32} & \text{in} \quad \mathbb{R} \\ \beta_2(3) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

- Since the classical Euler number $E_m = 2^m E_m(1/2) = 0$ when m is odd, the 2-adic value $\beta_2(m) = 0$ for every odd integer m.
- The value $\beta(2) = G = 0.9159655\cdots$ is known as *Catalan's constant*; it is not known whether G is irrational, but its 2-adic analogue $\beta_2(2) = G_2$ was shown to be irrational by Calegari in 2004.

Beta value examples

For $n = 0, 1, 2$ our expansion yields

$$\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)\binom{2m}{m}} = \begin{cases} \beta(1) = \pi/4 & \text{in} \quad \mathbb{R} \\ \beta_2(1) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

$$\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)\binom{2m}{m}} \left(H_{2m+1} - \frac{H_m}{2} \right) = \begin{cases} \beta(2) = G & \text{in} \quad \mathbb{R} \\ \beta_2(2) = G_2 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

$$\sum_{m=0}^{\infty} \frac{2^{m-2}}{(2m+1)\binom{2m}{m}} \left(\left(H_{2m+1} - \frac{H_m}{2} \right)^2 + \left(H_{2m+1}^{(2)} - \frac{H_m^{(2)}}{4} \right) \right) = \begin{cases} \beta(3) = \frac{\pi^3}{32} & \text{in} \quad \mathbb{R} \\ \beta_2(3) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

- Since the classical Euler number $E_m = 2^m E_m(1/2) = 0$ when m is odd, the 2-adic value $\beta_2(m) = 0$ for every odd integer m.

- The value $\beta(2) = G = 0.9159655\cdots$ is known as *Catalan's constant*; it is not known whether G is irrational, but its 2-adic analogue $\beta_2(2) = G_2$ was shown to be irrational by Calegari in 2004.

- As known to Dirichlet, you get a rational multiple of a power of π from $\zeta(2k)$ or from $\beta(2k+1)$. Yet the only one of the remaining (real) values whose arithmetic nature has been determined to date is Apéry's 1978 proof of the irrationality of $\zeta(3)$. We have had more success determining irrationality of their 2-adic and 3-adic analogues!

Derivatives of the beta function

Differentiating $\beta_2(s)$ at $s = 0$ yields

$$\sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{j=0}^{m} \binom{m}{j} (-1)^{j+1} \log_2(2j+1) = \begin{cases} \log(\frac{\Gamma(1/4)}{2\Gamma(3/4)}) & \text{in} \quad \mathbb{R}, \\ 2G_{2,1}(1/4) & \text{in} \quad \mathbb{Q}_2, \end{cases}$$

where

$$G_{2,1}(a) = \frac{\partial}{\partial s} \zeta_{2,1}(s,a) \Big|_{s=0}$$

denotes the Diamond 2-adic log gamma function.

Derivatives of the beta function

Differentiating $\beta_2(s)$ at $s = 0$ yields

$$\sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{j=0}^{m} \binom{m}{j} (-1)^{j+1} \log_2(2j+1) = \begin{cases} \log(\frac{\Gamma(1/4)}{2\Gamma(3/4)}) & \text{in} \quad \mathbb{R}, \\ 2G_{2,1}(1/4) & \text{in} \quad \mathbb{Q}_2, \end{cases}$$

where

$$G_{2,1}(a) = \frac{\partial}{\partial s} \zeta_{2,1}(s,a) \Big|_{s=0}$$

denotes the Diamond 2-adic log gamma function.

- No, that's not a mistake, the above series *does* converge 2-adically even though the outer summation has a high power of 2 in the *denominator*! Really, why would I lie?

Derivatives of the beta function

Differentiating $\beta_2(s)$ at $s = 0$ yields

$$\sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{j=0}^{m} \binom{m}{j} (-1)^{j+1} \log_2(2j+1) = \begin{cases} \log\left(\frac{\Gamma(1/4)}{2\Gamma(3/4)}\right) & \text{in} \quad \mathbb{R}, \\ 2G_{2,1}(1/4) & \text{in} \quad \mathbb{Q}_2, \end{cases}$$

where

$$G_{2,1}(a) = \frac{\partial}{\partial s} \zeta_{2,1}(s,a) \bigg|_{s=0}$$

denotes the Diamond 2-adic log gamma function.

- No, that's not a mistake, the above series *does* converge 2-adically even though the outer summation has a high power of 2 in the *denominator*! Really, why would I lie?
- Differentiating $\beta_2(s)$ at $s = -1$ yields

$$\sum_{m=0}^{\infty} \frac{1}{2^{m+2}} \sum_{j=0}^{m} \binom{m}{j} (-1)^{j+1} (2j+1) \log_2(2j+1) = \begin{cases} G/\pi & \text{in} \quad \mathbb{R}, \\ 0 & \text{in} \quad \mathbb{Q}_2 \end{cases}$$

where $G = \beta(2)$ is Catalan's constant.

Derivatives of the beta function

Differentiating $\beta_2(s)$ at $s = 0$ yields

$$\sum_{m=0}^{\infty} \frac{1}{2^{m+1}} \sum_{j=0}^{m} \binom{m}{j} (-1)^{j+1} \log_2(2j+1) = \begin{cases} \log(\frac{\Gamma(1/4)}{2\Gamma(3/4)}) & \text{in} \quad \mathbb{R}, \\ 2G_{2,1}(1/4) & \text{in} \quad \mathbb{Q}_2, \end{cases}$$

where

$$G_{2,1}(a) = \frac{\partial}{\partial s} \zeta_{2,1}(s,a) \bigg|_{s=0}$$

denotes the Diamond 2-adic log gamma function.

- No, that's not a mistake, the above series *does* converge 2-adically even though the outer summation has a high power of 2 in the *denominator*! Really, why would I lie?
- Differentiating $\beta_2(s)$ at $s = -1$ yields

$$\sum_{m=0}^{\infty} \frac{1}{2^{m+2}} \sum_{j=0}^{m} \binom{m}{j} (-1)^{j+1} (2j+1) \log_2(2j+1) = \begin{cases} G/\pi & \text{in} \quad \mathbb{R}, \\ 0 & \text{in} \quad \mathbb{Q}_2 \end{cases}$$

  where $G = \beta(2)$ is Catalan's constant.
- It's easy to explain why $\beta_2'(-1) = 0$. And I can give a decent heuristic explanation why a p-adic analogue of π should be zero that impresses people in bars. But I don't really know what it means that the real constant G/π should be correlated to a 2-adic value of zero.

Questions

Questions

- I also applied this construction to *modified zeta functions*, and I obtained the identity

$$\log_p a - \psi_p(a) = \sum_{m=0}^{\infty} \frac{(-1)^m m! b_{m+1}}{a(a+1)\cdots(a+m)}, \qquad \text{where} \qquad \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n$$

generates the *Bernoulli numbers of the second kind* b_n. The terms of the sequence are rational when $a \in \mathbb{Q}$ but the (real) sum is conjecturally not a period. Does this series have different meaning than the other examples?

Questions

- I also applied this construction to *modified zeta functions*, and I obtained the identity

$$\log_p a - \psi_p(a) = \sum_{m=0}^{\infty} \frac{(-1)^m m! b_{m+1}}{a(a+1)\cdots(a+m)}, \qquad \text{where} \qquad \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n$$

  generates the *Bernoulli numbers of the second kind* b_n. The terms of the sequence are rational when $a \in \mathbb{Q}$ but the (real) sum is conjecturally not a period. Does this series have different meaning than the other examples?

- Do these series linking real periods with their p-adic counterparts imply any relation in their respective natures? Or suggest a workable definition of a "p-adic period"?

Questions

- I also applied this construction to *modified zeta functions*, and I obtained the identity

$$\log_p a - \psi_p(a) = \sum_{m=0}^{\infty} \frac{(-1)^m m! b_{m+1}}{a(a+1)\cdots(a+m)}, \qquad \text{where} \qquad \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n$$

  generates the *Bernoulli numbers of the second kind* b_n. The terms of the sequence are rational when $a \in \mathbb{Q}$ but the (real) sum is conjecturally not a period. Does this series have different meaning than the other examples?

- Do these series linking real periods with their p-adic counterparts imply any relation in their respective natures? Or suggest a workable definition of a "p-adic period"?

- It looks like some p-adic things might not have real / complex analogues. For example, limit formulas, and the reflection formula for $\zeta_{p,r}(s,a)$. For another example, the identity $-2\xi_{2,2}(s, \frac{1}{2}) = s\xi_{2,1}(s+1, \frac{1}{2})$ for $s \in \mathbb{Z}_2^\times$, and a few others I found. I would like to prove / classify / explain such relations.

Questions

- I also applied this construction to *modified zeta functions*, and I obtained the identity

$$\log_p a - \psi_p(a) = \sum_{m=0}^{\infty} \frac{(-1)^m m! b_{m+1}}{a(a+1)\cdots(a+m)}, \qquad \text{where} \qquad \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n$$

  generates the *Bernoulli numbers of the second kind* b_n. The terms of the sequence are rational when $a \in \mathbb{Q}$ but the (real) sum is conjecturally not a period. Does this series have different meaning than the other examples?

- Do these series linking real periods with their p-adic counterparts imply any relation in their respective natures? Or suggest a workable definition of a "p-adic period"?

- It looks like some p-adic things might not have real / complex analogues. For example, limit formulas, and the reflection formula for $\zeta_{p,r}(s,a)$. For another example, the identity $-2\xi_{2,2}(s, \frac{1}{2}) = s\xi_{2,1}(s+1, \frac{1}{2})$ for $s \in \mathbb{Z}_2^{\times}$, and a few others I found. I would like to prove / classify / explain such relations.

- It would be nice to prove that the p-adic analogue of $\zeta(3)$ is *nonzero* for every prime p. Maybe these series can be used to show $L_p(3, \chi^0) \neq 0$. It's time to put an end to the scandal.

Questions

- I also applied this construction to *modified zeta functions*, and I obtained the identity

$$\log_p a - \psi_p(a) = \sum_{m=0}^{\infty} \frac{(-1)^m m! b_{m+1}}{a(a+1)\cdots(a+m)}, \qquad \text{where} \qquad \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n$$

  generates the *Bernoulli numbers of the second kind* b_n. The terms of the sequence are rational when $a \in \mathbb{Q}$ but the (real) sum is conjecturally not a period. Does this series have different meaning than the other examples?

- Do these series linking real periods with their p-adic counterparts imply any relation in their respective natures? Or suggest a workable definition of a "p-adic period"?

- It looks like some p-adic things might not have real / complex analogues. For example, limit formulas, and the reflection formula for $\zeta_{p,r}(s,a)$. For another example, the identity $-2\xi_{2,2}(s,\frac{1}{2}) = s\xi_{2,1}(s+1,\frac{1}{2})$ for $s \in \mathbb{Z}_2^\times$, and a few others I found. I would like to prove / classify / explain such relations.

- It would be nice to prove that the p-adic analogue of $\zeta(3)$ is *nonzero* for every prime p. Maybe these series can be used to show $L_p(3,\chi^0) \neq 0$. It's time to put an end to the scandal.

- Suppose that I could apply this construction to Barnes multiple zeta functions $\zeta_{p,r}(s,a)$ for $r \geq 2$. Differentiating at $s = 0$ gives p-adic multiple log gamma functions whose values might be used to compute Gross - Stark units. If we could compute Stark units *simultaneously* with Gross-Stark units, that would be really cool.

Thanks!

Preprints and videos can are available at `http://youngp.people.cofc.edu`