Differential descent obstructions

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Notations

Some stuff we will need.......

- k is a field of characteristic $p \geq 0$.
- K is a function field in one variable over k.
- We fix a separating element t of K/k and we will consider the derivation

$$\delta = d/dt : K \to K.$$

- Ω_K is the set of places of K and S is a finite subset of Ω_K of "bad places".
- for $v \in \Omega_K$, we consider the completion K_v and its ring of integers $\mathcal{O}_v$.
- $\mathcal{O}_S = \{x \in K : v(x) \geq 0, \text{ for all } v \notin S\}$ is the set of S-integers of K.

Main Question

We consider an affine variety X over K.

- $K[X]$ is the coordinate ring of X.
- We will consider adelic points

$$(x_v) \in \prod_{v \notin S} X(\mathcal{O}_v) \times \prod_{v \in S} X(K_v).$$

Main question

Under what conditions can we guarantee that

$$(x_v) \in X(\mathcal{O}_S)$$

that is, (x_v) will be global ?

Our approach : descent obstructions......

Motivation 1 : Artin-Schreier obstructions

- $p > 0$ and s is a power of p. The s-Artin-Schreier operator $\Phi_s(x) = x^s - x$.
- The exact sequence $0 \longrightarrow \mathbf{F}_s \longrightarrow \mathbf{G}_a \xrightarrow{x^s - x} \mathbf{G}_a \longrightarrow 0$ of fppf (or étale) sheaves on Spec $K[X]$ and on Spec K gives rise to group isomorphisms

$$H^1(\operatorname{Spec} K[X], \mathbf{F}_a) \cong \frac{K[X]}{\Phi_s(K[X])} \quad \text{and} \quad H^1(\operatorname{Spec} K, \mathbf{F}_s) \cong \frac{K}{\Phi_s(K)}.$$

Definition

An s-Artin-Schreier torsor over X is the image of an element of $H^1(\operatorname{Spec} K[X], \mathbf{F}_s)$ under

$$H^1(\operatorname{Spec} K[X], \mathbf{F}_s) \longrightarrow H^1(X, \mathbf{F}_s).$$

Theorem (D. Harari and J. F. Voloch, 2013)

If (x_v) is unobstructed by all s-Artin-Schreier torsors over X, for all s, then (x_v) is global.

Motivation 2 : Additive differential obstructions

The derivation $\delta = d/dt$ on K is additive....

- Define formal “cohomology groups"

$$H^1(K, \delta) := \frac{K}{\delta(K)} \quad \text{and} \quad H^1(K_v, \delta) := \frac{K_v}{\delta(K_v)}.$$

Definition

An additive differential torsor over X is given by a differential equation

$$\delta(z) = F, \quad F \in K[X].$$

An adelic point (x_v) is unobstructed by $F \in K[X]$ if there exist $(z_v) \in K_v$ and $c \in K$ such that

$$\delta(z_v) = F(x_v) + c, \quad \text{for all } v \in \Omega_K.$$

Theorem (J. F. Voloch, 2014)

If (x_v) is unobstructed by all additive differential torsors over X, then (x_v) is global.

Formalizing differential descent obstructions

- A differential scheme is a scheme X together with a global vector field $D \in Der_k(X)$.
- $K\{X\}$ is the ring of differential regular functions on X. $(\mathrm{Spec}\, K\{X\}, \delta)$ is a differential scheme.
- A morphism $(X, D) \to (X', D')$ of differential schemes is a morphism of schemes $X \to X'$ such that the sheaf homomorphism $\mathcal{O}_{X'} \to \mathcal{O}_X$ is differential.
- A differential fppf cover of (X, D) is a family $\{(X_i, D_i) \to (X, D)\}$ such that the cover $\{X_i \to X\}$ is fppf. This defines a differential fppf topology.

Theorem

The exact sequence $0 \longrightarrow \mathbf{G}_a^\delta \longrightarrow \mathbf{G}_a \xrightarrow{\delta} \mathbf{G}_a \longrightarrow 0$ *of differential fppf sheaves on* $\mathrm{Spec}\, K\{X\}$ *and on* $\mathrm{Spec}\, K$ *gives rise to group isomorphisms*

$$H^1(\mathrm{Spec}\, K\{X\}, \mathbf{G}_a^\delta) \cong \frac{K\{X\}}{\delta(K\{X\}} \quad \text{and} \quad H^1(\mathrm{Spec}\, K, \mathbf{G}_a^\delta) \cong \frac{K}{\delta(K)}.$$

Voloch's differential torsor $\delta(z) = F$, $F \in K[X]$, *corresponds to*

$$[F] \in H^1(\mathrm{Spec}\, K\{X\}, \mathbf{G}_a^\delta).$$

Multiplicative differential descent obstructions

- dlog is the log derivative $\mathrm{dlog}(x) = \delta(x)/x$.
- We also have an exact sequence $0 \longrightarrow \mathbf{G}_m^\delta \longrightarrow \mathbf{G}_m \xrightarrow{\mathrm{dlog}} \mathbf{G}_a \longrightarrow 0$ of differential fppf sheaves on $\operatorname{Spec} K\{X\}$ and on $\operatorname{Spec} K$. It induces :

$$H^1(\operatorname{Spec} K, \mathbf{G}_m^\delta) \cong \frac{K}{\mathrm{dlog}(K^\times)} \quad \text{and} \quad \frac{K\{X\}}{\delta(K\{X\}^\times)} \hookrightarrow H^1(\operatorname{Spec} K\{X\}, \mathbf{G}_m^\delta).$$

- An adelic point is unobstructed by $\delta(z)/z = F$, $F \in K[X]$, if there exist $(z_v) \in K_v^\times$ and $c \in K$ such that

$$\delta(z_v)/z_v = F(x_v) + c, \quad \text{for all } v \in \Omega_K.$$

Theorem

If (x_v) is unobstructed by all multiplicative differential torsors over X, then (x_v) is global.

By functoriality, it suffices to prove the result when $X = \mathbf{A}^1$.....

THANK YOU !
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