Differential descent obstructions

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Some stuff we will need.......

- $k$ is a field of characteristic $p \geq 0$.
- $K$ is a function field in one variable over $k$.
- We fix a separating element $t$ of $K/k$ and we will consider the derivation
  \[ \delta = \frac{d}{dt} : K \to K. \]
- $\Omega_K$ is the set of places of $K$ and $S$ is a finite subset of $\Omega_K$ of “bad places”.
- For $v \in \Omega_K$, we consider the completion $K_v$ and its ring of integers $\mathcal{O}_v$.
- $\mathcal{O}_S = \{ x \in K : v(x) \geq 0, \text{ for all } v \notin S \}$ is the set of $S$-integers of $K$. 
**Main Question**

We consider an affine variety $X$ over $K$.

- $K[X]$ is the coordinate ring of $X$.
- We will consider adelic points

$$\left(x_v\right) \in \prod_{v \notin S} X(\mathcal{O}_v) \times \prod_{v \in S} X(K_v).$$

**Main question**

Under what conditions can we guarantee that

$$\left(x_v\right) \in X(\mathcal{O}_S)$$

that is, $(x_v)$ will be global?

Our approach: descent obstructions......
Motivation 1: Artin-Schreier obstructions

- $p > 0$ and $s$ is a power of $p$. The $s$-Artin-Schreier operator $\Phi_s(x) = x^s - x$.

- The exact sequence $0 \rightarrow \mathbb{F}_s \rightarrow \mathbb{G}_a \xrightarrow{x^s-x} \mathbb{G}_a \rightarrow 0$ of fppf (or étale) sheaves on $\text{Spec } K[X]$ and on $\text{Spec } K$ gives rise to group isomorphisms

$$H^1(\text{Spec } K[X], \mathbb{F}_s) \cong \frac{K[X]}{\Phi_s(K[X])} \quad \text{and} \quad H^1(\text{Spec } K, \mathbb{F}_s) \cong \frac{K}{\Phi_s(K)}.$$

**Definition**

An $s$-Artin-Schreier torsor over $X$ is the image of an element of $H^1(\text{Spec } K[X], \mathbb{F}_s)$ under

$$H^1(\text{Spec } K[X], \mathbb{F}_s) \rightarrow H^1(X, \mathbb{F}_s).$$

**Theorem (D. Harari and J. F. Voloch, 2013)**

*If $(x_v)$ is unobstructed by all $s$-Artin-Schreier torsors over $X$, for all $s$, then $(x_v)$ is global.*
**Motivation 2: Additive differential obstructions**

The derivation $\delta = d/dt$ on $K$ is additive.

- Define formal “cohomology groups”

\[ H^1(K, \delta) := \frac{K}{\delta(K)} \quad \text{and} \quad H^1(K_v, \delta) := \frac{K_v}{\delta(K_v)}. \]

**Definition**

An additive differential torsor over $X$ is given by a differential equation

\[ \delta(z) = F, \quad F \in K[X]. \]

An adelic point $(x_v)$ is unobstructed by $F \in K[X]$ if there exist $(z_v) \in K_v$ and $c \in K$ such that

\[ \delta(z_v) = F(x_v) + c, \quad \text{for all } v \in \Omega_K. \]

**Theorem (J. F. Voloch, 2014)**

*If $(x_v)$ is unobstructed by all additive differential torsors over $X$, then $(x_v)$ is global.*
A differential scheme is a scheme $X$ together with a global vector field $D \in Der_k(X)$.

$K\{X\}$ is the ring of differential regular functions on $X$. $(\text{Spec } K\{X\}, \delta)$ is a differential scheme.

A morphism $(X, D) \to (X', D')$ of differential schemes is a morphism of schemes $X \to X'$ such that the sheaf homomorphism $\mathcal{O}_{X'} \to \mathcal{O}_X$ is differential.

A differential fppf cover of $(X, D)$ is a family $\{(X_i, D_i) \to (X, D)\}$ such that the cover $\{X_i \to X\}$ is fppf. This defines a differential fppf topology.

**Theorem**

The exact sequence $0 \to G^\delta_a \to G_a \xrightarrow{\delta} G_a \to 0$ of differential fppf sheaves on $\text{Spec } K\{X\}$ and on $\text{Spec } K$ gives rise to group isomorphisms

$$H^1(\text{Spec } K\{X\}, G^\delta_a) \cong \frac{K\{X\}}{\delta(K\{X\})} \quad \text{and} \quad H^1(\text{Spec } K, G^\delta_a) \cong \frac{K}{\delta(K)}.$$

Voloch's differential torsor $\delta(z) = F$, $F \in K[X]$, corresponds to

$$[F] \in H^1(\text{Spec } K\{X\}, G^\delta_a).$$
Multiplicative differential descent obstructions

- dlog is the log derivative \( \text{dlog}(x) = \delta(x)/x \).

- We also have an exact sequence \( 0 \to G_{\delta m} \to G_m \xrightarrow{\text{dlog}} G_a \to 0 \) of differential fppf sheaves on \( \text{Spec } K\{X\} \) and on \( \text{Spec } K \). It induces:

\[
H^1(\text{Spec } K, G_{\delta m}) \cong \frac{K}{\text{dlog}(K^\times)} \quad \text{and} \quad \frac{K\{X\}}{\delta(K\{X\}^\times)} \hookrightarrow H^1(\text{Spec } K\{X\}, G_{\delta m}).
\]

- An adelic point is unobstructed by \( \delta(z)/z = F, F \in K[X] \), if there exist \( (z_v) \in K_v^\times \) and \( c \in K \) such that

\[
\delta(z_v)/z_v = F(x_v) + c, \quad \text{for all } v \in \Omega_K.
\]

**Theorem**

*If \( (x_v) \) is unobstructed by all multiplicative differential torsors over \( X \), then \( (x_v) \) is global.*

By functoriality, it suffices to prove the result when \( X = \mathbb{A}^1 \).
THANK YOU!
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