Adventures in Solving Certain Families of Diophantine Equations

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Smith College

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History

$\sim$250 A.D. Diophantus lived in Alexandria.

'Here lies Diophantus,' the wonder behold.
Through art algebraic, the stone tells how old:
'God gave him his boyhood one-sixth of his life,
One twelfth more as youth while whiskers grew rife;
And then yet one-seventh ere marriage begun;
In five years there came a bouncing new son.
Alas, the dear child of master and sage
After attaining half the measure of his father's
life chill fate took him.
After consoling his fate by the science of numbers for four years,
he ended his life.'

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$$\begin{array}{lcl} 1 + A = x^k & & A = x^k - 1 \\ 1 + B = y^k & \iff & B = y^k - 1 \\ 1 + AB = z^k & & AB = z^k - 1, \end{array}$$

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$$(X^k - 1)(Y^k - 1) = (Z^k - 1)^2 \text{ with } X \neq \pm Y.$$

Theorem I

Let a, b, c, $k \in \mathbb{Z}^+$ with $k \geq 7$. Then the equation

$$(a^2cX^k - 1)(b^2cY^k - 1) = (abcZ^k - 1)^2$$

has no solutions in integers X, Y, $Z > 1$ with $a^2X^k \neq b^2Y^k$.

- Diophantine Approximations
  - Bennett (1998)
  - Standard results of continued fractions

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For small k, use continued fractions.

Theorem II

Let L, M, $N \in \mathbb{Z}^+$ with $N > 1$. Then the equation

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has no solutions with X, $Y \in \mathbb{Z}^+$ and $\gcd(NX, Y) = 1$.

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$$L_r(\gamma, \delta) = \begin{cases} \frac{\gamma^r - \delta^r}{\gamma - \delta} & \text{if } r \text{ is odd,} \\ \frac{\gamma^r - \delta^r}{\gamma^2 - \delta^2} & \text{if } r \text{ is even.} \end{cases}$$

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- r-defective Lehmer pair $(\gamma, \delta) \implies r \leq 30$.
- all r-defective Lehmer pairs are known for $6 < r \leq 30$, EXCEPT for when $r = 8, 10, 12$.

Theorem III

Let p be an odd prime and let N, α, β, $\gamma \in \mathbb{Z}$ with $N > 1$, $\alpha \geq 1$, and β, $\gamma \geq 0$. Then the equation

$$X^{2N} + 2^{2\alpha}5^{2\beta}p^{2\gamma} = Z^5$$

has no solutions with X, $Z \in \mathbb{Z}^+$ and $\gcd(X, Z) = 1$.

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- Modular Approach: Bennett-Skinner, 2004

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$\bar{f}$ a newform of wt 2 and level 2 $\Rightarrow\Leftarrow$

Theorem (Bennett & Skinner)

Let $x^7 + cy^7 = z^2$ *with* $c, x, y, z \in \mathbb{Z}$, $xy \neq \pm 1$,

- x, cy, *and* z *nonzero pairwise relatively prime,*
- $\forall$ *primes* q, $v_q(c) < 7$ *and* $v_2(cy^7) \geq 6$,
- $z \equiv 1 \pmod 4$,

Then $\exists$ *newform* f *of wt* 2 *and level*

$$N_7 = \begin{cases} 2\,\mathrm{rad}(c), & \text{if } v_2(c) = 0, \\ \mathrm{rad}(c)/2, & \text{if } v_2(c) = 6, \\ \mathrm{rad}(c), & \text{otherwise.} \end{cases}$$

Theorem (Bennett & Skinner)

Let $x^7 + cy^7 = z^2$ *with* $c, x, y, z \in \mathbb{Z}$, $xy \neq \pm 1$,

- x, cy, *and* z *nonzero pairwise relatively prime,*
- $\forall$ *primes* q, $v_q(c) < 7$ *and* $v_2(cy^7) \geq 6$,
- $z \equiv 1 \pmod 4$,

Then $\exists$ *newform* f *of wt* 2 *and level*

$$N_7 = \begin{cases} 2\,\mathrm{rad}(c), & \text{if } v_2(c) = 0, \\ \mathrm{rad}(c)/2, & \text{if } v_2(c) = 6, \\ \mathrm{rad}(c), & \text{otherwise.} \end{cases}$$

$$x^{2n} + 2^{2\alpha} 5^{2\beta} p^{2\gamma} = z^5$$

Extension of Theorem III

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Theorem (G- & Grundman)

The equation

$$X^{2N} + 4Y^2 = Z^5$$

has no solution with N, X, Y, $Z \in \mathbb{Z}^+$, $N > 1$, and $\gcd(X, Z) = 1$.

The adventure continues!