Refinement of Some Partition Inequalities

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Outline

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q-series Notation

q**-products:** $(a;q)_0 := 1$ and for $n \geq 1$,

$$
\begin{aligned}
(a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}) \\
(q;q)_n &:= (1-q)(1-q^2)\cdots(1-q^n) \;\; (*) \\
(a_1,\ldots,a_j;q)_n &:= (a_1;q)_n\cdots(a_j;q)_n \\
(a;q)_\infty &:= (1-a)(1-aq)(1-aq^2)\cdots \\
(a_1,\ldots,a_j;q)_\infty &:= (a_1;q)_\infty\cdots(a_j;q)_\infty
\end{aligned}
$$

The q-binomial theorem: if $|z|, |q| < 1$, then

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1}$$

Special Cases of the q-binomial theorem

Special Cases of the q-binomial theorem:

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \ |z| < 1, \ |q| < 1. \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{(-a)^n q^{n(n-1)/2}}{(q;q)_n} = (a;q)_\infty, \ |q| < 1. \tag{3}$$

Integer Partitions

Definition: A partition of a positive integer n is a way of writing n as a sum of positive integers, where order does not matter.

Example. The partitions of 5 are

$$
\begin{aligned}
&5\\
&4+1\\
&3+2\\
&3+1+1\\
&2+2+1\\
&2+1+1+1\\
&1+1+1+1+1
\end{aligned}
$$

The summands of a partition are called parts of the partition.

The number of partitions of n is given by the partition function $p(n)$.
For example, $p(5) = 7$.

Restricted Partition Functions, I

Some well known examples of restricted partition functions are $p_{\mathcal{O}}(n)$, the number of partitions of n into odd parts, and $p_{\mathcal{D}}(n)$, the number of partitions of n into distinct parts.

$$p_{\mathcal{O}}(5) = 3 \qquad (5,\ 3+1+1,\ 1+1+1+1+1),$$
$$p_{\mathcal{D}}(5) = 3 \qquad (5,\ 4+1,\ 3+2).$$

$$(P_{\mathcal{O}}(n) = P_{\mathcal{D}}(n),\ \forall n \in \mathbb{N})$$

Restricted Partition Functions, II

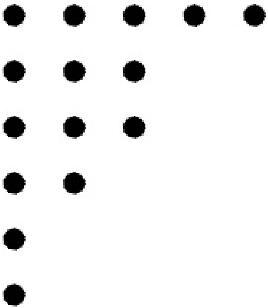
Let $p_{2,3,5}(n)$ denote the number of partitions of n into parts $\equiv 2, 3 \pmod 5$, and
$P^*(n)$ denote the number of partitions of n where each part from 1 to the largest part occurs at least twice.

$$p_{2,3,5}(10) = 4 \qquad \begin{array}{l} 2+2+2+2+2 \\ 3+3+2+2 \\ 7+3 \\ 8+2 \end{array}$$

$$P^*(10) = 4 \qquad \begin{array}{l} 1+1+1+1+1+1+1+1+1+1 \\ 2+2+1+1+1+1+1+1 \\ 2+2+2+1+1+1+1 \\ 2+2+2+2+1+1 \end{array}$$

$$(P_{2,3,5}(n) = P^*(n), \ \forall n \in \mathbb{N})$$

Ferrers Diagram, Durfee Square



Partition Generating Functions, I

Let S be any set of positive integers, finite or infinite. Then the generating function for $p_S(n)$, the number of partitions of the positive integer n with parts from S is

$$\sum_{n=0}^{\infty} p_S(n)q^n = \frac{1}{\prod_{a_i \in S} 1 - q^{a_i}}$$

$$= (1 + q^{a_1} + q^{2a_1} + q^{3a_1} + \dots)$$

$$\times (1 + q^{a_2} + q^{2a_2} + q^{3a_2} + \dots)$$

$$\times (1 + q^{a_3} + q^{2a_3} + q^{3a_3} + \dots) \dots$$

The generating function for $p_S^*(n)$, the number of partitions of the positive integer n with distinct parts from S is

$$\sum_{n=0}^{\infty} p_S^*(n)q^n = \prod_{a_i \in S} 1 + q^{a_i}$$

$$= (1 + q^{a_1})(1 + q^{a_2})(1 + q^{a_3}) \dots$$

Partition Generating Functions, II

Recall that $p(n)$ is the number of (unrestricted) partitions of n.

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(n)q^n &= \frac{1}{\prod_{k=1}^{\infty} 1 - q^k} = \frac{1}{(q;q)_\infty} \\
&= \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \\
&= \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k^2} \\
&= 1 + \sum_{k=1}^{\infty} \frac{q^k}{(q^k;q)_\infty} = 1 + \frac{1}{(q;q)_\infty} \sum_{k=1}^{\infty} (q;q)_{k-1} q^k
\end{aligned}
$$

Partition Generating Functions, III

Recall that $p_{\mathcal{D}}(n)$ is the number of partitions of n into distinct positive integers.

$$\sum_{n=0}^{\infty} p_{\mathcal{D}}(n) q^n = \prod_{k=1}^{\infty} (1 + q^k) = (-q; q)_\infty$$
$$= (1 + q)(1 + q^2)(1 + q^3) \dots$$

Partition Generating Functions, IV

Recall that $p_{2,3,5}(n)$ denote the number of partitions of n into parts $\equiv 2, 3 (\mod 5)$.

$$\sum_{n=0}^{\infty} p_{2,3,5}(n) q^n$$

$$= \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13})\dots}$$

$$= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = \frac{1}{(q^2,q^3;q^5)_\infty}$$

Partition Inequalities, I

Fact: For each positive integer n,

$$p_{1,4,5}(n) - p_{2,3,5}(n) \geq 0.$$

Alternatively, if the sequence $\{c_n\}$ is defined by

$$\sum_{n=0}^{\infty} c_n q^n = \frac{1}{(q, q^4; q^5)_\infty} - \frac{1}{(q^2, q^3; q^5)_\infty},$$

then $c_n \geq 0, \forall n \geq 0$.

Partition Inequalities, II

Proof.

By the Rogers-Ramanujan identities,

$$\frac{1}{(q,q^4;q^5)_\infty} - \frac{1}{(q^2,q^3;q^5)_\infty} = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} - \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k}$$

$$= \sum_{k=1}^{\infty} \frac{q^{k^2}(1-q^k)}{(q;q)_k}$$

$$= \sum_{k=1}^{\infty} \frac{q^{k^2}}{(q;q)_{k-1}}$$

□

Theorem (Berkovich and Garvan, 2005)

Suppose $L > 0$, and $1 < r < m - 1$. If the sequence $\{e_n\}$ is defined by

$$\sum_{n=0}^{\infty} e_n q^n = \frac{1}{(q, q^{m-1}; q^m)_L} - \frac{1}{(q^r, q^{m-r}; q^m)_L},$$

then

$$e_n \geq 0, \forall n \geq 0 \iff r \nmid m - r \text{ and } m - r \nmid r.$$

Theorem (Andrews, 2011)

If $L > 0$, and the sequence $\{f_n\}$ is defined by

$$\sum_{n=0}^{\infty} f_n q^n = \frac{1}{(q, q^5, q^6; q^8)_L} - \frac{1}{(q^2, q^3, q^7; q^8)_L},$$

then

$$f_n \geq 0, \forall n \geq 0.$$

Partition Inequalities, Variations and Extensions, III

Theorem (Berkovich and Grizzell, 2012)

For any $L > 0$, and any odd $y > 1$, the q-series expansion of

$$\frac{1}{(q, q^{y+2}, q^{2y}; q^{2y+2})_L} - \frac{1}{(q^2, q^y, q^{2y+1}; q^{2y+2})_L} = \sum_{n=0}^{\infty} a(L, y, n) q^n$$

has only non-negative coefficients. Furthermore, the coefficient $a(L, y, n)$ is 0 if and only if either

$$n \in \{2, 4, 6, \ldots, y+1\} \cup \{y\} \text{ or } (L, y, n) = (1, 3, 9).$$

Theorem (Berkovich and Grizzell, 2012)

For any $L > 0$, and any odd $y > 1$, and any x with $1 < x \leq y + 2$, the q-series expansion of

$$\frac{1}{(q, q^x, q^{2y}; q^{2y+2})_L} - \frac{1}{(q^2, q^y, q^{2y+1}; q^{2y+2})_L} = \sum_{n=0}^{\infty} a(L, x, y, n) q^n$$

has only non-negative coefficients. Furthermore, the coefficient $a(L, y, n)$ is 0 if and only if either ....

Theorem (Berkovich and Grizzell, 2013)

For any octuple of positive integers $(L, m, x, y, z, r, R, \rho)$*, the q-series expansion of*

$$\frac{1}{(q^x, q^y, q^z, q^{rx+Ry+\rho z}; q^m)_L} - \frac{1}{(q^{rx}, q^{Ry}, q^{\rho z}, q^{x+y+z}; q^m)_L} = \sum_{n=0}^{\infty} a(L, x, y, z, r, R, \rho, n) q^n$$

has only non-negative coefficients.

Partition Inequalities, Variations and Extensions, VI

Theorem (Berkovich and Grizzell, 2013)

For any positive integers m, n, y, and z, with $\gcd(n, y) = 1$, and integers K and L, with $K \geq L \geq 0$,

$$\frac{1}{(q^z;q^m)_K(q^{nyz};q^{nm})_L} - \frac{1}{(q^{yz};q^m)_K(q^{nz};q^{nm})_L} = \sum_{k=0}^{\infty} a(K, L, x, y, z, n, m, k)q^k$$

has only non-negative coefficients.

Partition Generating Functions that Track the Number of Parts

Let S be any set of positive integers, finite or infinite. Then the generating function for $p_S(m,n)$, the number of partitions of the positive integer n with exactly m parts from S is

$$\sum_{n=0}^{\infty} p_S(m,n)s^m q^n = \frac{1}{\prod_{a_i\in S} 1 - sq^{a_i}}$$

$$= (1 + sq^{a_1} + s^2q^{2a_1} + s^3q^{3a_1} + \dots)$$

$$\times (1 + sq^{a_2} + s^2q^{2a_2} + s^3q^{3a_2} + \dots)$$

$$\times (1 + sq^{a_3} + s^2q^{2a_3} + s^3q^{3a_3} + \dots)\dots$$

The generating function for $p_S^*(n)$, the number of partitions of the positive integer n with distinct parts from S is

$$\sum_{n=0}^{\infty} p_S^*(n)q^n = \prod_{a_i\in S} 1 + sq^{a_i}$$

$$= (1 + sq^{a_1})(1 + sq^{a_2})(1 + sq^{a_3})\dots$$

Q. If the polynomials $\{f_n(s)\}$ are defined by

$$\sum_{n=0}^{\infty} f_n(s)q^n = \frac{1}{(sq, sq^4; q^5)_\infty} - \frac{1}{(sq^2, sq^3; q^5)_\infty},$$

are there situations where the coefficients in $f_n(s)$ are all non-negative?

Experimental Output

n	$f_n(s)$
1	s
2	$-s + s^2$
3	$-s + s^3$
4	$s - s^2 + s^4$
5	s^5
6	$s - s^2 + s^6$
7	$-s + s^2 - s^3 + s^4 + s^7$
8	$-s + s^2 - s^4 + s^5 + s^8$
9	$s - s^2 + s^6 + s^9$
10	$s^7 + s^{10}$

Experimental Output

11	$s - s^2 + s^3 - s^4 + s^6 + s^8 + s^{11}$
12	$-s + 2s^2 - s^3 + s^4 - s^5 + s^7 + s^9 + s^{12}$
13	$-s + s^2 + s^5 - s^6 + s^7 + s^8 + s^{10} + s^{13}$
14	$s - 2s^2 + s^3 + s^6 - s^7 + s^8 + s^9 + s^{11} + s^{14}$
15	$s^7 + s^9 + s^{10} + s^{12} + s^{15}$
16	$s - 2s^2 + 2s^3 - s^4 + s^6 + s^8 + s^{10} + s^{11} + s^{13} + s^{16}$
17	$-s + 2s^2 - 3s^3 + 3s^4 - s^5 + s^7 + 2s^9 + s^{11} + s^{12} + s^{14} + s^{17}$
18	$-s + 2s^2 - s^3 + 2s^5 - 2s^6 + s^7 + s^8 + 2s^{10} + s^{12} + s^{13} + s^{15} + s^{18}$
19	$s - 2s^2 + 2s^3 - s^4 + 2s^6 - s^7 + s^8 + s^9 + s^{10} + 2s^{11} + s^{13} + s^{14} + s^{16} + s^{19}$
20	$s^5 + s^7 + s^9 + s^{10} + s^{11} + 2s^{12} + s^{14} + s^{15} + s^{17} + s^{20}$

Experimentation, II

5	s^5
10	$s^{10} + s^7$
15	$s^{15} + s^{12} + s^{10} + s^9 + s^7$
20	$s^{20} + s^{17} + s^{15} + s^{14} + 2s^{12} + s^{11} + s^{10} + s^9 + s^7 + s^5$
25	$s^{25} + s^{22} + s^{20} + s^{19} + 2s^{17} + s^{16} + 2s^{15} + 2s^{14} + s^{13}$ $+ 3s^{12} + s^{11} + 2s^{10} + 2s^9 + 2s^7 + s^5$
30	$s^{30} + s^{27} + s^{25} + s^{24} + 2s^{22} + s^{21} + 2s^{20} + 2s^{19} + s^{18}$ $+ 4s^{17} + 2s^{16} + 3s^{15} + 4s^{14} + s^{13} + 5s^{12} + 2s^{11} + 2s^{10}$ $+ 3s^9 + 3s^7 + s^5$
35	$s^{35} + s^{32} + s^{30} + s^{29} + 2s^{27} + s^{26} + 2s^{25} + 2s^{24} + s^{23}$ $+ 4s^{22} + 2s^{21} + 4s^{20} + 5s^{19} + 2s^{18} + 7s^{17} + 4s^{16} + 5s^{15}$ $+ 7s^{14} + 2s^{13} + 7s^{12} + 4s^{11} + 3s^{10} + 5s^9 + 4s^7 + s^5$

First Theorem

Theorem (Mc L. 2015)

Let $M \geq 5$ be a positive integer, and let a and b be integers such that $1 \leq a < b < M/2$ and $\gcd(a, M) = \gcd(b, M) = 1$. Define the integers $c(m, n)$ by

$$\frac{1}{(sq^a, sq^{M-a}; q^M)_\infty} - \frac{1}{(sq^b, sq^{M-b}; q^M)_\infty} := \sum_{m,n \geq 0} c(m, n) s^m q^n. \quad (4)$$

(i) Then $c(m, Mn) \geq 0$ for all integers $m \geq 0, n \geq 0$.
(ii) If, in addition, M is even, then $c(m, Mn + M/2) \geq 0$ for all integers $m \geq 0, n \geq 0$.

Partitions Interpretation

Corollary

Let M, a and b be as in Theorem 7. Let
$p_{a,M,m}(n) = \#$ *partitions of n into exactly m parts, each* $\equiv \pm a \pmod{M}$,
and let
$p_{b,M,m}(n) = \#$ *partitions of n into exactly m parts, each* $\equiv \pm b \pmod{M}$.

Then

(i) $p_{a,M,m}(nM) \geq p_{b,M,m}(nM)$ *for all integers* $n \geq 1$*, and all integers* m, $1 \leq m \leq Mn$.

(ii) If M is even, then $p_{a,M,m}(nM + M/2) \geq p_{b,M,m}(nM + M/2)$ *for all integers* $n \geq 0$*, and integers m with* $1 \leq m \leq Mn + M/2$.

Proof of First Theorem

Proof.

We recall a special case of the q-binomial theorem:

$$\frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}. \tag{5}$$

Hence

$$\frac{1}{(sq^a, sq^{M-a}; q^M)_\infty} - \frac{1}{(sq^b, sq^{M-b}; q^M)_\infty}$$
$$= \sum_{j,k\geq 0} \frac{s^{j+k} q^{a(j-k)+kM}}{(q^M;q^M)_j (q^M;q^M)_k} - \sum_{j,k\geq 0} \frac{s^{j+k} q^{b(j-k)+kM}}{(q^M;q^M)_j (q^M;q^M)_k} \tag{6}$$

Set $j+k =: m$, so that $j = m-k$ and the right side of (6) becomes

□

Proof, Continued

Proof Continued.

$$\sum_{m\geq 0} s^m \sum_{k=0}^{m} \frac{q^{a(m-2k)+kM} - q^{b(m-2k)+kM}}{(q^M;q^M)_{m-k}(q^M;q^M)_k} \qquad (7)$$

Next, we restrict the values of k so that when the inner sum is expanded as a power series, it contains only those powers of q whose exponents are multiples of M
(so that the series multiplying s^m is $\sum_{n=0}^{\infty} c(m, Mn)q^{Mn}$).

Since $\gcd(a, M) = \gcd(b, M) = 1$, this means restricting k so that $M|(m-2k)$.

If m is even, then $k = m/2$ is such a value, and $q^{a(m-2k)+km} - q^{b(m-2k)+km} = 0$ in this case.

Hence we need only consider those k in the intervals $0 \leq k < m/2$ and $m/2 < k \leq m$ satisfying $m - 2k \equiv 0 (\mod M)$. □

Proof, Continued

Proof Continued.

$$s^m \left(\sum_{0 \le k < m/2} + \sum_{m/2 < k \le m} \right) \frac{q^{a(m-2k)+kM} - q^{b(m-2k)+kM}}{(q^M;q^M)_{m-k}(q^M;q^M)_k}$$

Note that
(1) every k' in the upper interval may be expressed as $k' = m - k$, for some k in the lower interval;
(2) every k in the lower interval can be similarly matched with a k' in the upper interval;

$$(3) \;\; m - 2k \equiv 0 (\mod M) \iff m - 2(m-k) \equiv 0 (\mod M);$$

(4) the denominators of the summands remain invariant under the transformation $k \leftrightarrow m - k$.

□

Proof, Continued

Proof Continued.

$$\sum_{m,n\geq 0} c(m,Mn)s^m q^{Mn} = \sum_{m\geq 0} s^m \sum_{\substack{0\leq k<m/2 \\ M|m-2k}} \frac{\begin{array}{l} q^{a(m-2k)+kM} - q^{b(m-2k)+kM} \\ \quad + q^{-a(m-2k)+(m-k)M} - q^{-b(m-2k)+(m-k)M} \end{array}}{(q^M;q^M)_{m-k}(q^M;q^M)_k}$$

$$= \sum_{m\geq 0} s^m \sum_{\substack{0\leq k<m/2 \\ M|m-2k}} \frac{q^{a(m-2k)+kM}(1-q^{(m-2k)(b-a)})(1-q^{(m-2k)(M-b-a)})}{(q^M;q^M)_{m-k}(q^M;q^M)_k}$$

□

Proof, Continued

Proof Continued.

Finally,

- $M|(m-2k)(b-a)$ and $M|(m-2k)(M-b-a)$;

- the conditions on a and b give that they are different multiples of M, each less than $(m-k)M$;

- the factors $(1-q^{(m-2k)(b-a)})$ and $(1-q^{(m-2k)(M-b-a)})$ are cancelled by two different factors in the q-product $(q^M;q^M)_{m-k}$;

- the remaining factors in the denominators may be expanded as geometric series with only non-negative coefficients, and the claim at (i) above follows;

- The claim at (ii) follows similarly, upon noting that

$$m-2k \equiv M/2 \pmod M \iff m-2(m-k) \equiv -M/2 \equiv M/2 \pmod M.$$

□

Second Theorem

Theorem (Mc L. 2015)

Let $M \geq 5$ be a positive integer, and let a and b be integers such that $1 \leq a < b < M/2$ and $\gcd(a, M) = \gcd(b, M) = 1$. Define the integers $d(m, n)$ by

$$(-sq^a, -sq^{M-a}; q^M)_\infty - (-sq^b, -sq^{M-b}; q^M)_\infty := \sum_{m,n\geq 0} d(m, n)s^m q^n. \quad (8)$$

(i) Then $d(m, Mn) \geq 0$ for all integers $m \geq 0, n \geq 0$.
(ii) If, in addition, M is even, then $d(m, Mn + M/2) \geq 0$ for all integers $m \geq 0, n \geq 0$.

Partitions Interpretation

Corollary

Let M, a and b be as in Theorem 9. Let

$p^*_{a,M,m}(n)$ *denote the number of partitions of n into exactly m distinct parts* $\equiv \pm a \pmod{M}$*, and let*

$p^*_{b,M,m}(n)$ *denote the number of partitions of n into exactly m distinct parts* $\equiv \pm b \pmod{M}$.

Then

(i) $p^*_{a,M,m}(nM) \geq p^*_{b,M,m}(nM)$ *for all integers* $n \geq 1$*, and all integers m,* $1 \leq m \leq Mn$.

(ii) If M is even, then $p^*_{a,M,m}(nM + M/2) \geq p^*_{b,M,m}(nM + M/2)$ *for all integers* $n \geq 0$*, and integers m with* $1 \leq m \leq Mn + M/2$.

Sketch of Proof

Sketch of Proof.

Recall

$$(-a;q)_\infty = \sum_{n=0}^{\infty} \frac{a^n q^{n(n-1)/2}}{(q;q)_n}. \tag{9}$$

The application of this to the infinite products leads to

$$\sum_{m,n\geq 0} d(m,n) s^m q^n = \sum_{m\geq 0} s^m \sum_{k=0}^{m} \frac{(q^{a(m-2k)+kM} - q^{b(m-2k)+kM}) q^{M[(m-k)(m-k-1)/2+k(k-1)/2]}}{(q^M;q^M)_{m-k}(q^M;q^M)_k}$$

The proof now follows similarly. □

No Finite Analogues

It does not appear that replacing "∞" with a positive integer L in Theorems 7 and 9 "works". In other words, if L is a positive integer, and

$$\frac{1}{(sq^a, sq^{M-a}; q^M)_L} - \frac{1}{(sq^b, sq^{M-b}; q^M)_L} := \sum_{m,n\geq 0} c(m,n) s^m q^n, \quad (10)$$

it does not appear to be the case that $c(m, Mn) \geq 0$ for all m and n, and likewise for the other theorem.

Perhaps there is some restricted version of such a theorem that is valid?

Injective Proofs

Let $M \geq 5$ and $1 \leq a < b < M/2$ be integers. For integers $1 \leq m \leq n$ let $p_{a,M,m}(n) = \#$ partitions of n into exactly m parts, each
$\equiv \pm a (\mod M)$,
and let
$p_{b,M,m}(n) = \#$ partitions of n into exactly m parts, each
$\equiv \pm b (\mod M)$.
Can you find an injection from the partitions counted by $p_{b,M,m}(Mn)$ to those counted by $p_{a,M,m}(Mn)$?
Mind Floss: (1) Consider the partition of kMb consisting kM parts of size b.
Find a partition of kMb into kM parts, where each part is $\equiv \pm a (\mod M)$.

(2) Consider the partition of $kM(M-b)$ consisting kM parts of size $M-b$.
Find a partition of $kM(M-b)$ into kM parts, where each part is $\equiv \pm a (\mod M)$.

The End.