On two arithmetic theta lifts

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(Unitary) symmetric spaces

- $K = \mathbb{Q}(\sqrt{-\Delta})$ imaginary quadratic field, $\quad K \subset \mathbb{C}$

  [For simplicity, assume $\Delta \neq 1, 2$ and $h_K = 1$ throughout this talk]
- $L =$ self-dual Hermitian o_K-lattice
- Assume $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ has signature $(n-1, 1)$ over $\mathbb{C} = o_K \otimes \mathbb{R}$

$$\begin{aligned} \mathbb{D}(L) &:= \{z \subset L_{\mathbb{R}} \mid \dim_{\mathbb{C}}(z) = 1, \text{ and negative definite}\} \\ &\simeq U^{n-1} \qquad \text{[open unit ball]} \end{aligned}$$

Admits action by $U(L_{\mathbb{R}})$ and $\Gamma_L = \mathrm{Aut}(L) \subset U(L_{\mathbb{R}})$

$$\Longrightarrow [\Gamma_L \backslash \mathbb{D}(L)] \qquad \text{example of Shimura variety}$$

Special cycles

For $y \in L$, let

$$\mathbb{D}_y := \{z \in \mathbb{D}(L) \mid z \perp y\} \hookrightarrow \mathbb{D}(L)$$

[Note $(y, y) \leq 0 \implies \mathbb{D}_y = \emptyset$; otherwise $\mathbb{D}_y$ has complex codimension 1]

$$Z(m) := \sum_{\substack{y \in L \\ (y,y)=m}} \mathbb{D}_y$$

Locally finite: i.e. for any $K \subset \mathbb{D}(L)$ compact, there are only finitely many $y \in L$ with $(y, y) = m$ and $\mathbb{D}_y \cap K \neq \emptyset$

Green function

- Suppose $Z \subset X$ is a codimension 1 cycle.

- A *Green function for* Z is a current $[\mathfrak{g}] \in D^{(0,0)}(X)$ s.t.

$$dd^c[\mathfrak{g}] \; + \; \delta_Z \; = \; [\omega]$$

  for some smooth $(1,1)$-form ω, where $\delta_Z =$ integrate along Z

- In particular $[\mathfrak{g}]$ is smooth on $X - Z$ with singularities along Z

Kudla's Green function $\Xi(m, v)$

- For $z \in \mathbb{D}(L) \;=\; \{\text{negative definite lines } z \subset L_{\mathbb{R}}\}$ and $y \in L_{\mathbb{R}}$

$$R(y,z) \;:=\; \frac{|(y,z)|^2}{(z,z)}, \qquad\qquad R(y,z) = 0 \iff z \in \mathbb{D}_y$$

- For $t \in \mathbb{R}_{>0}$,

$$\beta(t) \;:=\; \int_1^\infty e^{-rt} r^{-1} dr$$

- $\beta(t) = -\log(t) + O(1)$ as $t \to 0$

For a fixed parameter $v \in \mathbb{R}_{>0}$

$$\Xi(m,v)(z) \;:=\; \sum_{\substack{y \in L \\ (y,y)=m}} \beta_1(2\,\pi\, v\, R(y,z)), \qquad z \in \mathbb{D}(L) - Z(m)$$

Has log singularity along $Z(m)$.

Bruinier's Green function $\Phi(F_m)$

"Regularized theta lift" après Borcherds

$$\Phi(F_m)(z) \ := \ \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}}^{reg} F_m(\tau)\ \Theta_{\mathrm{Sieg}}(\tau, z)\ d\mu(\tau)$$

- Siegel theta function $\Theta_{\mathrm{Sieg}} : \mathfrak{H} \times \mathbb{D}(L) \to \mathbb{C}$
- Fix z; then $\Theta_{\mathrm{Sieg}}(\cdot, z)$ is a non-holomorphic mod form of weight $n-2$
- $F_m = m$'th Hejhal-Poincaré series of weight $2-n$
- regularization:

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}}^{reg} F_m(\tau)\,\Theta(\tau, z) d\mu(\tau) \ := \ \underset{s=0}{CT} \int_{\mathcal{F}} F_m(\tau)\,\Theta(\tau, z)\, v^{-s-2}\, du\, dv$$

[*N.B.:* We should actually take 'vector-valued' versions of these objects...]

Kudla-Millson:

For all $m \in \mathbb{Z}$,

$$dd^c[\Xi(m,v)] \; + \; \delta_{Z(m)} \; = \; \text{m'th Fourier coeff of } [\Theta_{KM}(\tau)]$$

where $[\Theta_{KM}(\tau)]$ is a modular form valued in $A^{1,1}(X)$

Bruinier-Funke: *On two geometric theta lifts*

For all $m > 0$

$$dd^c[\Phi(F_m)] \; + \; \delta_{Z(m)} \; = \; \text{m'th Fourier coeff of } \pi_{\mathrm{hol}}(\Theta_{KM})(\tau)$$

i.e. $dd^c\left(\Xi(m,v) - \Phi(F_m)\right)$ is the m'th F.C. of a modular form with **trivial holomorphic projection**.

Q: can we relate $\Xi(m,v)$ and $\Phi(F_m)$ directly?

Vector-valued C^∞ modular forms

- **Weil representation** $\rho\colon SL_2(\mathbb{Z}) \to W := \mathbb{C}[\partial^{-1}L/L]$

Define

$$A_k(\rho) \ := \ \left\{ f : \mathfrak{H} \to W \mid f \text{ is } C^\infty,\ f(\gamma\tau) = (c\tau + d)^k \rho(\gamma) f(\tau) \right\}$$

Shimura-Maaß lowering operator

$$\mathcal{L}\colon A_k(\rho) \to A_{k-2}(\rho), \qquad \mathcal{L}(f) \ = \ -2\ i\ v^2\ \frac{\partial f}{\partial \overline{\tau}}$$

Moderate and exponential growth forms

- $f \in A_k$ has **at worst moderate growth** if

$$f(u+iv) = O(v^n) \quad \text{for some } n \text{ [as } v \to \infty]$$

- $f \in A_k$ has **at worst exponential growth** if

$$f(u+iv) = O(e^{Cv}) \quad \text{for some } C \in \mathbb{R}$$

Define

- $M_k^{\mathrm{exp}}(\rho) \;=\; \{\text{exp growth forms}\}$
- $M_k^{\mathrm{mod}}(\rho) \;=\; \{\text{mod growth} + \text{a "constant term" condition}\}$

Theorem (Ehlen-S.)

Suppose $f \in M_k^{\mathrm{mod}}(\rho)$ *with* $k > 0$. *Then* $\exists$ *a* **unique**

$$G(\tau) \;=\; \sum_{m\in\mathbb{Q}} c_G(m,v)\; e^{2\pi i m\tau} \;\in\; M_{k+2}^{\exp}(\rho)$$

1. $\mathcal{L}(G) = f$
2. $\lim_{v\to\infty} c_G(m,v) = 0$ *for all* $m < 0$
3. *the 'constant' part of* $c_G(0,v)$ *vanishes*
4. G *has trivial "holomorphic projection"; i.e.*

$$\int^{reg} G(\tau)\,\overline{\phi(\tau)}\, v^{(k+2)/2} d\mu(\tau) = 0$$

for every cusp form $\phi \in S_{k+2}(\rho)$.

[Write $G = \mathcal{L}^\sharp(f)$.]

OK, but can we find $\mathcal{L}^\sharp(f)$ explicitly?

For $w \in \mathbb{R}_{>0}$ and $\tau \in \mathfrak{H}$, define **cutoff function**

$$\sigma_w(\tau) := \begin{cases} 1, & \text{if } \Im(\tau) \geq w \\ 0, & \text{otherwise} \end{cases}$$

Truncated Poincaré series

For any $m \in \mathbb{Z}$ and weight $k \in \mathbb{Z}$, define

$$\mathcal{P}_{m,w}(\tau) := \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} (c\tau + d)^{-k} e^{-2\pi i m \gamma\tau} \sigma_w(\gamma\tau)$$

- $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right),\ n \in \mathbb{Z}\}$
- Converges absolutely, (highly discontinuous!) modular function of weight k.
- Also has vector-valued variant

Theorem (Ehlen-S.)

Suppose $f \in M_k^{mod}(\rho)$ with $k > 0$ and

$$G(\tau) \ = \ \mathcal{L}^\sharp(f)(\tau) \ = \ \sum_m \ c_G(m,v) \ e^{2\pi i m\tau}, \qquad v = \Im(\tau)$$

Then for $m > 0$

$$c_G(m,v) \ = \ \int^{reg} f(\tau') \ \left(F_m(\tau') - \mathcal{P}_{m,v}(\tau')\right) d\mu(\tau')$$

and for $m \leq 0$

$$c_G(m,v) = -\int^{reg} f(\tau') \, \mathcal{P}_{m,v}(\tau') \ d\mu(\tau')$$

In particular, these regularized integrals exist.
(Here $\mathcal{P}_{m,v}$ and F_m are of weight $-k$)

Idea of proof: unfold Poincaré series and use Stokes' theorem.

Back to problem

Theorem (Ehlen-S.)

For $m > 0$,

$$\Xi(m,v)(z) \; - \; \Phi(F_m)(z) = \int^{\mathrm{reg}} \left(\mathcal{P}_{m,v}(\tau') - F_m(\tau')\right) \Theta_{\mathrm{Sieg}}(\tau', z) d\mu(\tau')$$

and for $m \leq 0$

$$\Xi(m,v)(z) \; = \; \int^{\mathrm{reg}} \mathcal{P}_{m,v}(\tau') \, \Theta_{\mathrm{Sieg}}(\tau', z) d\mu(\tau') \; + \; \delta_{m,0} \log v$$

Corollary

Suppose $n > 2$. Let $q := e^{2\pi i\tau}$,

$$F(\tau, z) \;:=\; \sum_{m<0} \Xi(m,v)(z)\; q^m \;+\; (\Xi(0,v)(z) \;-\; \log v) \\ \qquad\qquad +\; \sum_{m>0} [\Xi(m,v)(z) - \Phi(F_m)(z)]\; q^m$$

is (pointwise in z) a modular form with **trivial holomorphic projection.**

Proof: $F(\tau, z) = -\mathcal{L}^\sharp(\Theta(\cdot, z))$

Applications in arithmetic geometry (unitary case)

Integral models:

- $\mathcal{M} \to \mathrm{Spec}(o_K)$ (PEL moduli problem)
- $\mathcal{Z}(m) \to \mathcal{M}$ divisor (also via moduli problem)

Howard constructs $\mathcal{M}^* =$ canonical toroidal compactification of $\mathcal{M}$

Arithmetic classes

(in $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M}^*) =$ Burgos-Kramer-Kühn Chow group)

$$\widehat{\mathcal{Z}}^{\mathrm{Kud}}(m, v) := \left(\mathcal{Z}^*(m) + (\text{bdy terms}),\ \Xi(m, v)\right)$$

$$\widehat{\mathcal{Z}}^{\mathrm{Bru}}(m) := \left(\mathcal{Z}^*(m) + (\text{bdy terms}),\ \Phi(F_m)\right)$$

where $\mathcal{Z}^*(m) =$ Zariski closure of $Z(m)$ in $\mathcal{M}^*$

- bdy terms computed by Howard and Bruinier-Howard-Yang

arithmetic Theta functions:

$$\widehat{\Theta}^{\mathsf{Kud}}(\tau) = \sum_{m\in\mathbb{Z}} \widehat{\mathcal{Z}}^{\mathsf{Kud}}(m,v)\,q^m \quad \text{and} \quad \widehat{\Theta}^{\mathsf{Bru}}(\tau) = \sum_{m\geq 0} \widehat{\mathcal{Z}}^{\mathsf{Bru}}(m)\,q^m$$

Theorem (Ehlen-S.)

For $n > 2$, $\widehat{\Theta}^{\mathsf{Kud}}(\tau) - \widehat{\Theta}^{\mathsf{Bru}}(\tau)$ is a modular form (of weight n) with trivial holomorphic projection.

Idea of proof: for $m > 0$, write

$$\widehat{\mathcal{Z}}^{\mathsf{Kud}}(m,v) - \widehat{\mathcal{Z}}^{\mathsf{Bru}}(m) \;=\; ((\text{bdy terms}),\; \Xi(m,v) - \Phi(F_m))$$

Can compute

- bdy terms of the form $\mathcal{L}^\sharp(\Theta_\Lambda)$ for some sig $(n-2,0)$ lattice Λ
- $\Xi(m,v) - \Phi(F_m)$ is m'th F.C. of $-\mathcal{L}^\sharp(\Theta_{\mathrm{Sieg}})$

Kudla's arithmetic height conjecture

- ω = tautological bundle on $\mathcal{M}^*$, can be equipped with a metric

$$\rightsquigarrow \ \widehat{\omega} \ \in \ \widehat{\mathrm{Pic}}(\mathcal{M}^*)$$

- Arithmetic height pairing:

$$[\,\bullet \ : \ \widehat{\omega}^{n-1}] \ : \ \widehat{\mathrm{CH}}^1_{\mathbb{C}}(\mathcal{M}^*) \ \to \ \mathbb{C}$$

Write $[\widehat{\Theta}^{\mathrm{Kud}}(\tau) : \widehat{\omega}^{n-1}] := \sum[\widehat{\mathcal{Z}}^{\mathrm{Kud}}(m,v) : \widehat{\omega}^{n-1}]q^m$

Kudla's conjecture (rough form)

$$[\widehat{\Theta}^{\mathrm{Kud}}(\tau) : \widehat{\omega}^{n-1}] \ = \ C \cdot E'(\tau, s_0) \ + \ \text{(vertical and bdy correction terms)}$$

where $E'(\tau, s_0)$ is the derivative of an Eisenstein series $E(\tau, s)$ at $s_0 = n-1$.

Can isolate the "non-holomorphic part" and boundary parts of this conjecture by taking difference of theta functions:

Theorem (Ehlen-S.)

For $n > 2$,

$$[\widehat{\Theta}^{\mathrm{Kud}}(\tau) - \widehat{\Theta}^{\mathrm{Bru}}(\tau) : \widehat{\omega}^{n-1}] \;=\; C \cdot E'(\tau, s_0) \;+\; 4\pi \sum_{[B_\Lambda]} \mathcal{L}^{\sharp}\, \Theta_\Lambda(\tau) \cdot [\widehat{B_\Lambda}]$$

Sketch of proof: e.g. for $m > 0$

$$[\widehat{\mathcal{Z}}^{\mathrm{Kud}}(m,v) - \widehat{\mathcal{Z}}^{\mathrm{Bru}}(m) : \widehat{\omega}^{n-1}] \;\approx\; (\mathrm{bdy}) + \int_{\mathcal{M}^*(\mathbb{C})} (\Xi(m,v) - \Phi(F_m))\; d\Omega$$

Sketch of proof (cont.):

$$\int_{\mathcal{M}^*(\mathbb{C})} (\Xi(m,v) - \Phi(F_m))\ d\Omega \ = \ \int^{reg} (\mathcal{P}_{m,v} - F_m)\ I(\Theta_{\mathrm{Sieg}})\ d\mu(\tau')$$
$$= \text{m'th F.C. of } -\mathcal{L}^{\sharp} I(\Theta_{\mathrm{Sieg}})$$

where

$$I(\Theta)(\tau') \ = \ \int_{\mathcal{M}^*(\mathbb{C})} \Theta_{\mathrm{Sieg}}(\tau', z) d\Omega(z) \ \overset{S.W.}{\approx} \ E_0(\tau')$$

where $E_0(\tau')$ is a holomorphic Eisenstein series and S.W. = Siegel-Weil formula

Easy to check:

- $\mathcal{L}(E'(\cdot, s_0)) \ = \ -2\ E_0$
- $E'(\tau', s_0)$ has trivial principal part and holomorphic projection.

$\therefore$ by uniqueness characterizing $\mathcal{L}^{\sharp}$, have

$$\mathcal{L}^{\sharp} I(\Theta) \approx E'(\cdot, s_0)$$