Differential obstructions to rational points on projective curves

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Introduction

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Question: Can we describe $X(K)$? Is $X(K) \neq \emptyset$?

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Question: Can we describe $X(K)$? Is $X(K) \neq \emptyset$?

Attempt to answer: Describe $X(K_v)$, for each place v of K.

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Question: If $\prod_v X(K_v) \neq \emptyset$, is $X(K)$ nonempty?

Descent obstructions

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$$X(K) \longrightarrow \prod_v X(K_v)$$

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$$X(K)^\pi = \left\{ (x_v) : ([\pi](x_v)) = (c), \text{ for some } c \in H^1(K, G) \right\}.$$

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We say that (x_v) is *unobstructed* by π.

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$$X(K) \subset \bigcap_{\pi \in H^1(X,G)} X(K)^\pi.$$

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- Say that (x_v) is unobstructed by $\delta z = F$ if there exist $c \in K$ and $(z_v) \in \prod_v K_v$ such that $\delta z_v = F(x_v) + c$.

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Theorem

$$X(K) = \bigcap_{\pi} X(K)^{\pi},$$

where the intersection runs over all torsors π given by $\delta z = F$, for $F \in K[X]$.

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Natural questions:

- What is going on? Can we unify the multiplicative and additive versions?

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<u>Natural questions:</u>

- What is going on? Can we unify the multiplicative and additive versions?
- What happens if X is not affine?

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We can now talk about the *differential fppf site* on X, denoted X_{dfl}.

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- $\mathbf{G}_a^{\delta} = \ker(\mathbf{G}_a \xrightarrow{\text{der}} \mathbf{G}_a)$

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$$K[X] \xrightarrow{D} K[X] \to H^1(X, \mathbf{G}_a^\delta) \to H^1(X, \mathbf{G}_a) \xrightarrow{D^*} H^1(X, \mathbf{G}_a)$$

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$$K[X]^\times \xrightarrow{\mathrm{dlog}} K[X] \to H^1(X, \mathbf{G}_m^\delta) \to \mathrm{Pic}\, X \xrightarrow{\mathrm{dlog}^*} H^1(X, \mathbf{G}_a)$$

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Projective curves

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Voloch (2012):

Theorem

If X is a smooth projective curve of genus $g \geq 2$ over K and $\delta \in \operatorname{Der} K$ has nontrivial Kodaira-Spencer class in X, then $X(K)$ is described by differential descent obstructions.

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Idea:

- (Buium, Voloch) X^1 is affine.
- We have an open immersion $X \to X^1$.

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If X *is a smooth projective curve of genus* $g \geq 2$ *over* K*, then* $X(K) - X(K^{\delta})$ *is described by differential descent obstructions.*

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Idea:

- If X not hyperelliptic, then

$$X \to X^1 \to \operatorname{Spec} O_{X^1} \to \operatorname{Spec} S(H^0(X, \Omega_{X/K})) = \mathbf{A}^g$$

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Idea:

- If X hyperelliptic, $charK \neq 2$ and X is given by an equation $y^2 = f(x)$ with coefficients in K^δ, then look at the torsors

$$\pi_1 : \delta z = \frac{\delta(x)}{2y} \quad \text{and} \quad \pi_2 : \delta z = x\frac{\delta(x)}{2y}.$$

Further questions

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- What happens when X has genus 1?

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- What about higher dimensional projective varieties X?

Thank you!