A talk about "Euclidean algorithms and stuff."

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Multiquadratic Fields

An n-quadratic field, $n \geq 1$ is any degree 2^n field of the form

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The set $\{a_1, \ldots, a_n\}$ is called a *radicand list* for the field.

Norm-Euclidean Fields

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That is, K is norm-Euclidean if for any $a, b \in \mathcal{O}_K$, $b \neq 0$ there exist $q, r \in \mathcal{O}_K$ such that $a = qb + r$ and $|N_{K/\mathbb{Q}}(r)| < |N_{K/\mathbb{Q}}(b)|$.

Euclidean Quadratic Fields

(Dirichlet 1842, Wantzel 1848, Davenport 1948 et.al.)
The norm-Euclidean quadratic fields have been fully classified.
They are the fields $\mathbb{Q}(\sqrt{a})$ where a is in the set

$$\{-11, -7, -3, -2, -1, 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73\}.$$

(Clark, 1994) The field $K := \mathbb{Q}(\sqrt{69})$ is not norm-Euclidean, but is Euclidean under the multiplicative function f given by

$$f\left(10 + 3\frac{1+\sqrt{69}}{2}\right) = 26$$

and $f(p) = |N(p)|$ for any other prime $p \in \mathcal{O}_K$.

Euclidean Biquadratic Fields

Theorem (Lemmermeyer)

There are exactly 13 norm-Euclidean imaginary biquadratic fields; they are given by $\mathbb{Q}(\sqrt{a_1}, \sqrt{a_2})$ *where*

$$a_1 = -1,\ a_2 = 2, 3, 5, 7;$$
$$a_1 = -2,\ a_2 = -3, 5;$$
$$a_1 = -3,\ a_2 = 2, 5, -7, -11, 17, -19;$$
$$a_1 = -7,\ a_2 = 5.$$

Multiquadratic Number Fields

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We will look at the imaginary multiquadratic fields first

Kuroda's Class Number Formula

Theorem (Lemmermeyer)

Let K/k be a V_4 extension of number fields. Then Kuroda's class number formula holds:

$$h(K) = 2^{d-\kappa-2-\nu} q(K) h_1 h_2 h_3 / h_k^2.$$

- d *is the number of infinite places ramified in* K/k
- κ *is the* $\mathbb{Z}$*-rank of the unit group of* $\mathcal{O}_k$
- $\nu \in \{0, 1\}$
- h_k *is the class number of* k
- h_1, h_2, h_3 *are the class numbers of the intermediate fields* $k \subset k_1, k_2, k_2 \subset K$

Class Number of Triquadratic Fields

(To appear in the Journal of Number Theory)

Theorem (Feaver)

There are 17 imaginary triquadratic fields with class number 1. These are the fields with radicand lists $\{a_1, a_2, a_3\}$ *given in the following table:*

$\{a_1, a_2, a_3\}$	$\{a_1, a_2, a_3\}$	$\{a_1, a_2, a_3\}$	$\{a_1, a_2, a_3\}$
$\{-1, 2, 3\}$	$\{-1, 3, 5\}$	$\{-1, 7, 19\}$	$\{-3, -7, -15\}$
$\{-1, 2, 5\}$	$\{-1, 3, 7\}$	$\{-1, 7, 91\}$	$\{-3, -11, -6\}$
$\{-1, 2, 11\}$	$\{-1, 3, 11\}$	$\{-2, -3, -7\}$	$\{-3, -11, -19\}$
$\{-1, 5, 7\}$	$\{-1, 3, 19\}$	$\{-2, -3, 5\}$	$\{-3, -11, 17\}$
		$\{-2, -7, 5\}$	

And, when $n \geq 4$, there are no imaginary n-quadratic fields of class number 1.

Norm-Euclidean Fields

Theorem (Lemmermeyer)

$$\begin{array}{ccc} K & & \mathfrak{B}^n \\ \Big| {\scriptstyle n} & & \Big| \\ k & & \mathfrak{p} \end{array}$$

If K is norm-Euclidean, then for any $\alpha \in \mathcal{O}_k \setminus \mathfrak{p}$, there exists $b \in \mathcal{O}_k$ such that

- $b \equiv \alpha^n \bmod \mathfrak{p}$,
- $b = N_{K/k}\delta$ *for some* $\delta \in \mathcal{O}_K$ *and*
- $|N_{k/\mathbb{Q}} b| < |N_{k/\mathbb{Q}} \mathfrak{p}|$.

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Let $\alpha = \frac{1+\sqrt{-11}}{2}$;

$$b \equiv \alpha^4 = \frac{7 - 5\sqrt{-11}}{2} \equiv \frac{-1 - \sqrt{-11}}{2} \bmod 2.$$

Norm-Euclideanity of Triquadratic Fields

$|N_{k/\mathbb{Q}}\mathfrak{p}| = 4$ so the only choice for b with $|N_{k/\mathbb{Q}}b| < |N_{k/\mathbb{Q}}\mathfrak{p}|$ is $b = \frac{-1-\sqrt{-11}}{2}$; which gives $N_{k/\mathbb{Q}}b = 3$.

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Applying the theorem, $K = \mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{-11})$ is not norm-Euclidean.

Theorem (Feaver)

Let K/k be a finite, abelian, normal extension of number fields of relative degree n. Let $\mathfrak{p} \subset \mathcal{O}_k$ be a non-zero prime ideal and let e denote the ramification index of $\mathfrak{p}$ in $\mathcal{O}_K$. If K is norm-Euclidean, then for any $\alpha, \beta \in \mathcal{O}_k \setminus \mathfrak{p}$ with $\beta \equiv \alpha^n \bmod \mathfrak{p}$, there exists $b \in \mathcal{O}_k$ such that

$$b = N_{K/k}\delta \text{ for some } \delta \in \mathcal{O}_K,$$

$$b \equiv \beta \bmod \mathfrak{p} \text{ and}$$

$$|N_{k/\mathbb{Q}} b| < |N_{k/\mathbb{Q}} \mathfrak{p}|^{n/e}.$$

A field which is not norm-Euclidean:

$$K = \mathbb{Q}(\sqrt{-1}, \sqrt{-7}, \sqrt{-91}),$$

$$k = \mathbb{Q}(\sqrt{-91})$$

$$\mathfrak{p} = 2$$

$$\alpha = \frac{1}{2}(1 + \sqrt{-91})$$

$$b \bmod \mathfrak{p} = \frac{1}{2}(\pm 1 \pm \sqrt{-91})$$

Not norm-Euclidean:
$\{-1, 7, 19\}$, $\{-1, 7, 91\}$, $\{-1, 2, 11\}$, $\{-1, 3, 11\}$, $\{-1, 3, 19\}$

The End!