When shifted primes are practical

Victor Guo [1] Andreas Weingartner [2]

[1] University of Missouri

[2] Southern Utah University

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The sequence of practical numbers:

$$1,\ 2,\ 4,\ 6,\ 8,\ 12,\ 16,\ 18,\ 20,\ 24,\ 28,\ 30,\ 32,\ 36,\ 40, \ldots$$

Characterization of practical numbers

Stewart (1954) and Sierpinski (1955): An integer $n \geq 2$ with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \quad p_1 < p_2 < \ldots < p_k$, is practical if and only if

$$p_j \leq 1 + \sigma\left(\prod_{1 \leq i < j} p_i^{\alpha_i}\right) \qquad (1 \leq j \leq k),$$

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Example: $1204 = 2^2 \cdot 7 \cdot 43$ is practical because

$$2 \leq 1 + \sigma(1) = 2, \quad 7 \leq 1 + \sigma(4) = 8, \quad 43 \leq 1 + \sigma(4 \cdot 7) = 57.$$

Analogies with prime numbers

Practical numbers: 1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, ...

Prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

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[Prime Number Theorem] (W., 2015):
The number of practical numbers below x is asymptotic to $\frac{c\,x}{\log x}$ for some positive constant c.

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- Is $p + a$ prime infinitely often?

Some primes get practical when shift happens

Shifted primes $p-1$: $1,\ 2,\ 4,\ 6,\ 10,\ 12,\ 16,\ 18,\ 22,\ 28,\ 30, \ldots$

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Question: For how many primes p is $p+a$ is practical?

Heuristic: If the events "p is prime" and "$p+a$ is practical" are roughly independent, we would expect that the number of primes $p \le x$ with $p+a$ practical is about $\dfrac{x}{(\log x)^2}$.

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Definition: Let $P_a(x)$ denote the number of primes $p \le x$ with $p+a$ practical.

Theorem (Guo, W.): Let a be a fixed odd integer. We have

$$\frac{x}{(\log x)^{5.769}} \ll P_a(x) \ll \frac{x}{(\log x)^{1.086}}.$$

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These bounds for $P_a(x)$ still hold if the sequence of practical numbers is replaced by the sequence of

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- integers $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ with the property

$$p_j \leq \theta(p_1^{\alpha_1} \cdots p_{j-1}^{\alpha_{j-1}}) \qquad (1 \leq j \leq k),$$

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where θ is an arithmetic function which satisfies

$$\max(2, n) \le \theta(n) \ll n \exp\Big((\log\log 3n)^{17.427}\Big).$$

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Thus

$$P_a(x) \le H(x, \sqrt{x}, \sqrt{x}\log x, a) \ll \frac{x}{(\log x)^{1.086}}.$$

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Thus the number of pairs (q, k) such that qk is practical and $qk + a$ is prime is

$$\gg \frac{\sqrt{x}}{\log x} \cdot \frac{\sqrt{x}}{\log x} = \frac{x}{(\log x)^2}.$$

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Thus

$$P_a(x) \gg \frac{x}{(\log x)^2} \frac{1}{(\log x)^{2c}} > \frac{x}{(\log x)^{5.769}}.$$

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Thank You!