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Author(s): Bart Goddard

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FINITE EXPONENTIAL SERIES AND NEWMAN POLYNOMIALS

BART GODDARD

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ABSTRACT. A Newman polynomial is a sum of powers of z, with constant term 1. The Newman polynomial of four terms whose minimum modulus on the unit circle is as large as possible is found by examining the expression

$$f(4) = \sup_{x_1<\cdots<x_4} \inf_{\alpha\in\mathfrak{R}} \left|\sum_{j=1}^{4} e^{ix_j\alpha}\right|$$

and determining an extremal system $(x_1, \ldots, x_4)$ using a technique that reduces the problem to a finite search.

1. INTRODUCTION

Let $P(z) = \sum_{j=1}^{n} a_j z^{r_j}$ be a complex polynomial. Erdös [1] and Littlewood [2] asked several questions concerning the minimum modulus of $P(z)$ on the unit circle, under various restrictions of the coefficients a_j, e.g., $|a_j| = 1$ for $j = 1, 2, \ldots, n$. If we insist that $a_j = 1$ for $j = 2, \ldots, n$ and $r_1 = 0$, then $P(z)$ is a Newman polynomial, as defined by Campbell, Ferguson, and Forcade [3]. Many other authors have investigated the minimum modulus of Newman polynomials, most notably Smyth [4] and Boyd [5].

Rudolfer and Hayman [7] ask for information about

$$f(n) = \sup_{x_1<x_2<\cdots<x_n} \inf_{\alpha\in\mathfrak{R}} \left|\sum_{j=1}^{n} e^{i\alpha x_j}\right| .$$

If $x_1 = r_1$, $x_2 = r_2$, $\ldots$, $x_n = r_n$ are natural numbers, we have

$$f(n) = \sup_{r_1<r_2<\cdots<r_n} \min_{|z|=1} |P(z)| .$$

The purpose of this paper is to calculate $f(4)$ explicitly and, in the process, discover some examples of Newman polynomials with few terms, but large minimum modulus. $f(2)$ is trivially 0, and $f(3)$ is calculated in [3], being

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attained for the Newman polynomial $1+z^2+z^3$. We shall prove here that $f(4)$ is attained for $1+z^2+z^3+z^4$.

2. Preliminaries

For n a natural number, we define $F_n\colon \mathbb{R}^n \to \mathbb{R}$ via

$$F_n(x_1, \ldots, x_n) = \left|\sum_{j=1}^{n} e^{ix_j}\right|^2 . \tag{1}$$

Then we have $f(n) = \sup_{x_1<\cdots<x_n} \inf_{\alpha\in\mathfrak{R}} F_n(x_1\alpha, \ldots, x_n\alpha)^{1/2}$. It is easy to show that

$$F_n(x_1, \ldots, x_n) = n + 2\sum_{j<k} \cos(x_k - x_j), \tag{2}$$

and from this, that

$$F_n(x_1, x_2, \ldots, x_n) = F_n(x_1, x_n + x_1 - x_{n-1}, \ldots, x_n + x_1 - x_2, x_n). \tag{3}$$

The simplification

$$f(n) = \sup_{0=A_1<A_2<\cdots<A_n} \inf_{\alpha\in\mathfrak{R}} F_n(0, A_2\alpha, \ldots, A_n\alpha)^{1/2}$$

where $0=A_1<A_2<\cdots<A_n$ are nonnegative integers and $\gcd(A_2, A_3, \ldots, A_n) = 1$, is given as Theorem 1 of [3], or we may proceed as follows: It suffices to show that for every n-tuple $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\varepsilon > 0$ there is an n-tuple $(r_1, \ldots, r_n) \in \mathbb{Q}^n$ such that

$$\inf_{\alpha\in\mathfrak{R}} \left|\sum_{j=1}^{n} e^{i\alpha x_j}\right| \le \inf_{\alpha\in\mathfrak{R}} \left|\sum_{j=1}^{n} e^{i\alpha r_j}\right| + \varepsilon .$$

If each $x_j \in \mathbb{Q}$, we are done. Otherwise, by the simultaneous rational approximation theorem (Hardy and Wright [8, p. 170]) there are infinitely many solutions to the system of inequalities

$$\left|x_j - \frac{p_j}{q}\right| < \frac{1}{q^{(1+1/n)}}, \qquad j = 1, 2, \ldots, n.$$

Further, the function $h(x) = e^{ix}$ is continuous and periodic, hence uniformly continuous over $\mathbb{R}$, so there exists a $\delta > 0$ such that $|e^{ix} - e^{iy}| < \varepsilon/n$ whenever $|x-y| < \delta$. Let $[p_1/q, \ldots, p_n/q]$ be a solution to the inequalities with $q > [2\pi/\delta]^n$. Then $q^{1/n} > 2\pi/\delta$, and hence $\delta > 2\pi/q^{1/n}$. Then the function of α, $g(\alpha) = |\sum_{j=1}^{n} e^{i\alpha p_j/q}|$ has period $2\pi q$. Now for $0 \le \alpha \le 2\pi q$ we have

$$\left|\sum_{j=1}^{n} e^{i\alpha x_j} - \sum_{j=1}^{n} e^{i\alpha p_j/q}\right| \le \sum_{j=1}^{n} \left|e^{i\alpha x_j} - e^{i\alpha p_j/q}\right| \le \sum_{j=1}^{n} \frac{\varepsilon}{n} = \varepsilon$$

since $|\alpha x_j - \alpha p_j/q| = \alpha|x_j - p_j/q| < 2\pi q/q^{(1+1/n)} = 2\pi/q^{1/n} < \delta$.

So,

$$\inf_{\alpha\in[0,2\pi q]} \left|\sum_{j=1}^{n} e^{i\alpha x_j}\right| \le \inf_{a\in[0,2\pi q]} \left|\sum_{j=1}^{n} e^{i\alpha p_j/q}\right| + \varepsilon .$$

Then, since we are taking the infimum over a smaller set, we have

$$\begin{aligned}\inf_{\alpha\in\Re}\left|\sum_{j=1}^{n} e^{i\alpha x_j}\right| &\le \inf_{\alpha\in[0,2\pi q]}\left|\sum_{j=1}^{n} e^{i\alpha x_j}\right| \\ &\le \inf_{\alpha\in[0,2\pi q]}\left|\sum_{j=1}^{n} e^{i\alpha p_j/q}\right| + \varepsilon = \inf_{\alpha\in\Re}\left|\sum_{j=1}^{n} e^{i\alpha p_j/q}\right| + \varepsilon\end{aligned}$$

since $g(\alpha)$ has period $2\pi q$. Consequently, the following variation of (3) is considered:

(4) $F_n(0, A_2, A_3, \ldots, A_n) = F_n(0, A_n - A_{n-1}, A_n - A_{n-2}, \ldots A_n - A_2, A_n)$.

3. Lemmas

We will need the following lemmas. Recall from (1) and (2) that

$$F_4(x_1, x_2, x_3, x_4) = \left|\sum_{j=1}^{4} e^{ix_j}\right|^2 = 4 + 2\sum_{j<k}\cos(x_k - x_j).$$

Lemma 1. *Given distinct integers* (x_1, x_2, x_3, x_4), *there is a zero* (z_1, z_2, z_3, z_4) *of* F_4 *and a* $t_0 \in \mathbb{R}$ *such that*

(i) $x_2 t_0 = z_2$; $x_3 t_0 = z_3$; $x_4 t_0 = z_4$;
(ii) $|z_1 - x_1 t_0| \le \pi \gcd(x_4 - x_3, x_2 - x_1)/(|x_4 - x_3|)$; *and*
(iii) $(z_4 - z_3)$ *and* $(z_2 - z_1)$ *are odd multiples of* π.

Proof. Let $d = \gcd(x_4 - x_3, x_2 - x_1)$. Consider the linear Diophantine equation in l and k,

(5) $$2l(x_2 - x_1) + 2k(x_4 - x_3) = (x_1 - x_2) + (x_3 - x_4) + \beta d$$

where $\beta = 0$ or 1 is chosen so that the right-hand side is an even multiple of d. With β so chosen, (5) is solvable. Let $l = l_0$ and $k = k_0$ be a solution. Let $t_0 = (2l_0 + 1)\pi/(x_4 - x_3)$, $z_1 = t_0 x_2 + (2k_0 + 1)\pi$, $z_2 = x_2 t_0$, $z_3 = x_3 t_0$, and $z_4 = x_4 t_0$. Then it is easy to check that $(z_4 - z_3) = (2l_0 + 1)\pi$ and $(z_2 - z_1) = -(2k_0 + 1)\pi$. It remains to show (ii) is satisfied:

$$\begin{aligned}|z_1 - x_1 t_0| &= |t_0 x_2 + (2k_0 + 1)\pi - x_1 t_0| \\ &= |t_0(x_2 - x_1) + (2k_0 + 1)\pi| \\ &= \left|\frac{(2l_0 + 1)\pi}{x_4 - x_3}(x_2 - x_1) + (2k_0 + 1)\pi\right| \\ &= \frac{\pi}{|x_4 - x_3|}|(2l_0 + 1)(x_2 - x_1) + (2k_0 + 1)(x_4 - x_3)| \\ &= \frac{\pi}{|x_4 - x_3|}|\beta d| \le \frac{\pi d}{|x_4 - x_3|},\end{aligned}$$

which completes the proof.

Lemma 2. *Let* w *be a positive real number. Let* A_1, A_2, A_3 *be distinct natural numbers such that* $\gcd(A_1, A_2, A_3) = 1$ *and* $w(\gcd(A_1, A_2 - A_3)) \le A_1$. *Then*

$$\inf_{\alpha\in\Re} F_4(0, A_1\alpha, A_2\alpha, A_3\alpha) < (\pi/w)^2.$$

Proof. Let $d = (A_1, A_2 - A_3)$. Then we have $d/A_1 \leq 1/w$. For the point $(0, A_1, A_2, A_3)$ in $\mathbb{R}^4$, Lemma 1 gives a zero $(z_1, z_2, z_3, z_4) \in \mathbb{R}^4$ and an $\alpha_0 \in \mathbb{R}$ such that $0\alpha_0 = z_1$, $A_1\alpha_0 = z_2$, $A_2\alpha_0 = z_3$, and

$$|A_3\alpha_0 - z_4| < \frac{\pi \cdot \gcd(A_1, A_3 - A_2)}{A_1} = \frac{\pi d}{A_1} \leq \frac{\pi}{w}.$$

Further, $z_2 = (z_2 - z_1)$ and $(z_4 - z_3)$ are odd multiples of π, say $z_2 = (z_2 - z_1) = (2k+1)\pi$ and $(z_4 - z_3) = (2l+1)\pi$. Let $\gamma = A_3\alpha_0 - z_4$. Now we compute

$$\begin{aligned}\inf_{a\in\mathfrak{R}} F_4(0, A_1\alpha, A_2\alpha, A_3\alpha) &\leq F_4(0, A_1\alpha_0, A_2\alpha_0, A_3\alpha_0)\\ &= |1 + e^{iz_2} + e^{iz_3} + e^{iA_3\alpha_0}|^2\\ &= |1 + e^{i(2k+1)\pi} + e^{iz_4}(e^{i(z_3-z_4)} + e^{i(A_3\alpha - z_4)})|^2\\ &= |1 - 1 + e^{iz_4}(e^{-i(2l+1)\pi} + e^{i\gamma})|^2\\ &= |e^{iz_4}|^2| - 1 + e^{i\gamma}|^2 = 1 \cdot |e^{i\gamma/2} - e^{-i\gamma/2}|^2\\ &= 4\sin^2\left(\frac{\gamma}{2}\right) \leq 4\left(\frac{\gamma}{2}\right)^2 = \gamma^2 < \left(\frac{\pi}{w}\right)^2, \quad \text{as desired.}\end{aligned}$$

Lemma 3. *Let (a, b, c) be a triple of natural numbers such that*

(i) $a < b < c$,
(ii) $\gcd(a, b, c) = 1$,
(iii) $c - a < b$,
(iv) $\gcd(a, c - b) > a/4.18$,
(v) $\gcd(b, c - a) > b/4.18$,
(vi) $\gcd(c, b - a) > c/4.18$.

Then $(a, b, c) = (2, 3, 4)$ *or* $(4, 9, 10)$.

Proof. Since $\gcd(c, b - a)$ is a divisor of c, $\gcd(c, b - a)/c$ is a rational number of the form $1/m$ where m is a natural number. Then $1/m > 1/4.18$ from (vi). Whence $m < 4.18$. Since $b - a < c$, $\gcd(c, b - a) < c$, so $m \neq 1$. Therefore, the possible values of m are 2, 3, and 4.

If $m = 2$ then $b - a = c/2$, hence $2b - 2a = c$.

If $m = 3$ then $b - a = c/3$ or $2c/3$, so $3b - 3a = c$ or $3b - 3a = 2c$.

If $m = 4$ then $b - a = c/4$ or $3c/4$, so $4b - 4a = c$ or $4b - 4a = 3c$.

So (a, b, c) must satisfy one of the five Diophantine equations:

$$2b - 2a = c, \qquad 3b - 3a = c, \qquad 3b - 3a = 2c,$$
$$4b - 4a = c, \qquad 4b - 4a = 3c.$$

Similarly, using inequalities (v) and (iii), we have that (a, b, c) must satisfy one of the five Diophantine equations:

$$2c - 2a = b, \qquad 3c - 3a = b, \qquad 3c - 3a = 2b,$$
$$4c - 4a = b, \qquad 4c - 4a = 3b.$$

First suppose $k(b-a) = lc$ and $j(c-a) = ib$ for integers k, l, j, and i; that is, (a, b, c) satisfy one of the first five and one of the second five equations above. Then solving for b and c in terms of a yields

$$c = \left[\frac{kj + ki}{kj - li}\right] a, \tag{6}$$

$$b = \left[\frac{kj + li}{kj - li}\right] a. \tag{7}$$

Define t by $a = (kj - li)t$. Then from (6) and (7)

$$b = \frac{(kj + lj)}{(kj - li)}(kj - li)t = (kj + li)t \quad \text{and} \quad c = \frac{(kj + ki)}{(kj - li)}(kj - li)t = (kj + ki)t.$$

Then t is rational, say $t = p/q$ with p, q relatively prime integers. Then $p|a$, b, and c, whence $p = 1$. So $t = 1/q$ for some natural number q. Then $q|(kj - li)$, $(kj + lj)$, and $(kj + ki)$. Since $\gcd(a, b, c) = 1$, we must have $q = \gcd((kj - li), (kj + lj), (kj + ki))$. Thus there is only one solution to the Diophantine system

$$\begin{cases} k(b - a) = lc, \\ j(c - a) = ib, \end{cases}$$

with $(a, b, c) = 1$. Also, since $b < c$, we have $(kj + lj)t < (kj + ki)t$. So $lj < ki$. There are 25 possibilities for the tuple (k, l, j, i), corresponding to the 25 possible Diophantine systems. The following table lists those tuples with $lj < ki$, along with the value of q and the corresponding solution (a, b, c).

Table I

k, l, j, i	$kj - li$	$kj + lj$	$kj + ki$	q	(a, b, c)
2, 1, 3, 2	4	9	10	1	(4, 9, 10)
2, 1, 4, 3	5	12	14	1	(5, 12, 14)
3, 1, 2, 1	5	8	9	1	(5, 8, 9)
3, 1, 3, 2	7	12	15	1	(7, 12, 15)
3, 1, 4, 3	9	16	21	1	(9, 16, 21)
3, 2, 4, 3	6	20	21	1	(6, 20, 21)
4, 1, 2, 1	7	10	12	1	(7, 10, 12)
4, 1, 3, 1	11	15	16	1	(11, 15, 16)
4, 1, 3, 2	10	15	20	5	(2, 3, 4)
4, 1, 4, 3	13	20	28	1	(13, 20, 28)

Now it remains to eliminate most of the triples (a, b, c) in Table I, by showing that they violate one of the inequalities (iv)–(vi). Now $c - a < b$, so $c - b < a$; whence $a \nmid (c - b)$, so if a is a prime bigger than 4, we have

$$\frac{\gcd(a, c - b)}{a} = \frac{1}{a} \le \frac{1}{5} < \frac{1}{4.18},$$

so the triples $(5, 12, 14)$, $(5, 8, 9)$, $(7, 12, 15)$, $(7, 10, 12)$, $(11, 15, 16)$, and $(13, 20, 28)$ violate inequality (iv). This leaves only $(4, 9, 10)$, $(9, 16, 21)$, $(6, 20, 21)$, and $(2, 3, 4)$. But

$$\frac{\gcd(9, 21-16)}{9} = \frac{\gcd(9, 5)}{9} = \frac{1}{9} < \frac{1}{4.18}$$

and

$$\frac{\gcd(6, 21-20)}{6} = \frac{1}{6} < \frac{1}{4.18},$$

which violate (iv). This leaves $(4, 9, 10)$ and $(2, 3, 4)$ as claimed.

Lemma 4. $\inf_{\alpha\in\mathfrak{R}} F_4(0, 2\alpha, 3\alpha, 4\alpha) = 0.566\ldots$.

Proof. From (3) and the Chebyshev polynomials,

$$\begin{aligned} F_4(0, 2\alpha, 3\alpha, 4\alpha) &= 4 + 2(\cos 2\alpha + \cos 3\alpha + \cos 4\alpha + \cos\alpha + \cos 2\alpha + \cos\alpha) \\ &= 16\cos^4\alpha + 8\cos^3\alpha - 8\cos^2\alpha - 2\cos\alpha + 2. \end{aligned}$$

Then

$$\inf_{\alpha\in\mathfrak{R}} F_4(0, 2\alpha, 3\alpha, 4\alpha) = \min_{-1\le x\le 1}(16x^4 + 8x^3 - 8x^2 - 2x + 2).$$

It is a simple calculus exercise to show this last expression is equal to $0.566\ldots$ as desired.

Note that this is equivalent to saying

$$\min_{|z|=1}|1 + z^2 + z^3 + z^4| = (0.566\ldots)^{1/2} = 0.7524\ldots,$$

which appears in Table 1 of [5]. Thus we have a Newman polynomial of only four terms, with relatively large minimum modulus on the unit circle. The next theorem shows this result is best possible.

4. Main result

Theorem 1. $f(4) = 0.7524\ldots$.

Proof. Let $A_1 < A_2 < A_3$ be distinct positive integers with $(A_1, A_2, A_3) = 1$. From the functional relationship (3), we have

$$F_4(0, A_1\alpha, A_2\alpha, A_3\alpha) = F_4(0, (A_3 - A_2)\alpha, (A_3 - A_1)\alpha, A_3\alpha),$$

so if $A_3 - A_1 > A_2$ let $A_1' = A_3 - A_2$, $A_2' = A_3 - A_1$, and $A_3' = A_3$. Then $A_3' - A_1' = A_3 - (A_3 - A_2) = A_2 < A_3 - A_1 = A_2'$. So we may assume without loss of generality that $A_3 - A_1 \le A_2$.

If $A_3 - A_1 = A_2$, then exactly two of $\{A_1, A_2, A_3\}$ are odd, so exactly two of $\{e^{iA_1\pi}, e^{iA_2\pi}, e^{iA_3\pi}\}$ are equal to -1 and the other is equal to 1. Therefore,

$$\begin{aligned} \inf_{\alpha\in\mathfrak{R}} F_4(0, A_1\alpha, A_2\alpha, A_3\alpha) &\le F_4(0, A_1\pi, A_2\pi, A_3\pi) \\ &= |1 + e^{iA_1\pi} + e^{iA_2\pi} + e^{iA_3\pi}| = |1 + 1 - 1 - 1|^2 = 0 < 0.566\ldots. \end{aligned}$$

So we may assume $A_3 - A_1 < A_2$. Now if A_1, A_2, A_3 violate one of the inequalities (iv)–(vi) in Lemma 3, we have, by Lemma 2, with $w = 4.18$, that

$$\inf_{\alpha\in\mathfrak{R}} F_4(0, A_1\alpha, A_2\alpha, A_3\alpha) < \left[\frac{\pi}{4.18}\right]^2 < 0.566\ldots = \inf_{\alpha\in\mathfrak{R}} F_4(0, 2\alpha, 3\alpha, 4\alpha).$$

Therefore, to find $\sup_{0<A_1<A_2<A_3} \inf_{\alpha\in\mathfrak{R}} |F_4(0, A_1\alpha, A_2\alpha, A_3\alpha)|$, it suffices to look only at triples (A_1, A_2, A_3) that satisfy the hypotheses of Lemma 3. But this means we need only check $(2, 3, 4)$ and $(4, 9, 10)$. Now

$$\begin{aligned}\inf_{\alpha}|F_4(0, 4\alpha, 9\alpha, 10\alpha)| &\le F_4(0, 4(\tfrac{4}{10}), 9(\tfrac{4}{10}), 10(\tfrac{4}{10}))\\ &= 0.3758\ldots < 0.566\ldots.\end{aligned}$$

Therefore

$$\begin{aligned}f(A) &= \sup_{\substack{0<A_1<A_2<A_3\\ \gcd(A_1, A_2, A_3)=1}} \inf_{\alpha\in\mathfrak{R}} (F_4(0, A_1\alpha, A_2\alpha, A_3\alpha))^{1/2}\\ &= \inf_{\alpha\in\mathfrak{R}} (F_4(0, 2\alpha, 3\alpha, 4\alpha))^{1/2} = (0.5661\ldots)^{1/2} = 0.7524\ldots.\end{aligned}$$

5. Further results

In an effort to see how fast $f(n)$ grows (or see if it is, in fact, monotonic), we explicitly computed several examples to estimate the size of $f(5)$ and $f(6)$.

First, we generated all quadruples (A_1, A_2, A_3, A_4) of natural numbers with $\gcd(A_1, A_2, A_3, A_4) = 1$ and $0 < A_1 < A_2 < A_3 < A_4 \le 30$. For each quadruple, we computed the values of

$$F_5(0, A_1\alpha, A_2\alpha, A_3\alpha, A_4\alpha) \quad \text{for } \alpha = 0, 0.01, 0.02, \ldots, 3.15$$

and saved the smallest value. The largest of these came from the quadruple $(1, 2, 6, 9)$. The minimum value of $F_5(0, \alpha, 2\alpha, 6\alpha, 9\alpha)$ apparently occurs when $\alpha = \pi$ and gives the surprising value

$$\begin{aligned}F_5(0, \pi, 2\pi, 6\pi, 9\pi) &= |1 + e^{i\pi} + e^{i2\pi} + e^{i6\pi} + e^{i9\pi}|2\\ &= |1 - 1 + 1 + 1 - 1|^2 = 1,\end{aligned}$$

which also appears in Table 1 of [5]. Thus $f(5) \ge 1$. We did the same for $f(6)$. We checked all tuples $0 < A_1 < A_2 < A_3 < A_4 < A_5 \le 30$ and all values of α from 0 to π in increments of 0.001 and found that

$$\begin{aligned}f(6) &= \sup_{0<A_1<\cdots<A_5} \inf_{\alpha\in\mathfrak{R}} \left|1 + \sum_{j=1}^{5} e^{iA_j\alpha}\right|\\ &\ge \inf_{\alpha\in\mathfrak{R}} |1 + e^{i6\alpha} + e^{i9\alpha} + e^{i10\alpha} + e^{i17\alpha} + e^{i24\alpha}| \approx 1.1348\ldots,\end{aligned}$$

which is achieved when $\alpha \approx 2.45$.

Thus the Newman polynomials $1 + z + z^2 + z^6 + z^9$ and $1 + z^6 + z^9 + z^{10} + z^{17} + z^{24}$ have only five and six terms, but yet have minimum modulus on the unit circle larger than or equal to 1. In [5] Boyd shows that $f(n) > 1$ for $6 \le n \le 16$ and conjectures that $\log f(n)/\log n \to \alpha > 0$. It seems quite likely that $f(n)$ is at least monotonic.

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Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute, Indiana 47803-3999
E-mail address: goddard@nextwork.rose-hulman.edu