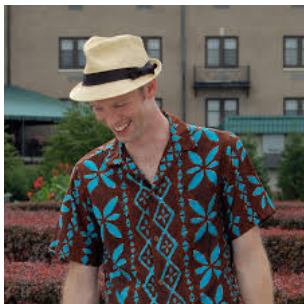


On the field of definition of a cubic rational function and its critical points

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- $K =$ field with characteristic 0
- $\overline{K} =$ algebraic closure of K
- $f, g \in \overline{K}(z)$ are *equivalent* if there exists a linear fractional transformation $\sigma \in \overline{K}$ such that $f = \sigma \circ g$.

A Case of the B. and M. Shapiro Conjecture



Theorem (Eremenko-Gabrielov)

If $f \in \mathbb{C}(z)$ is a rational function with only real critical points, then f is equivalent to a rational function with real coefficients.



- **Goldberg:** There are at most

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- **Eremenko and Gabrielov:** Using topological, combinatorial, and complex analytic techniques construct exactly $\rho(d)$ real rational functions with a given set of real critical points.

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- This leads to the following question:

Question:

Is there a truly elementary proof of the Eremenko and Gabrielov's result?

Corollary (Faber, T.)

Using only algebraic techniques, we show if $f \in \mathbb{C}(z)$ is a degree 3 rational function with only real critical points, then f is equivalent to a rational function with real coefficients.

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Note:

- The quadratic case is trivial over any field.
- If $f \in \overline{K}(z)$ has critical points $c_1, c_2 \in \mathbb{P}^1(K)$, $c_1 \neq \infty$, then either $f = \left(\frac{z-c_1}{z-c_2}\right)^2$ or $f = (z - c_1)^2$.

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- f has 4 distinct critical points.

Normal form for cubic

- We begin with a normal form for cubic functions. For $u \in \overline{K} \setminus \{-1, -2\}$, define

$$f_u(z) = \frac{z^2(z + u)}{(2u + 3)z - (u + 2)}. \quad (1)$$

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Lemma

A cubic rational function that is critical at 0, 1, and ∞ is equivalent to a unique f_u , and the fourth critical point is $\phi(u) = -u \frac{u+2}{2u+3}$.

Proposition

If $f_u \in \overline{K}(z)$ is equivalent to a rational function with K -coefficients, then $u \in K$.

Algebraic Condition

Definition

For a field K and rational function $\phi \in K(z)$, we say K is ϕ -**perfect** if the map $\phi : \mathbb{P}^1(K) \rightarrow \mathbb{P}^1(K)$ is surjective.

Theorem (Faber, T.)

Let K be a field of characteristic zero with algebraic closure \overline{K} . The following statements are equivalent:

- 1. Any cubic rational function $f \in \overline{K}(z)$ with K -rational critical points is equivalent to a rational function in $K(z)$.*
- 2. K is ϕ -perfect, where $\phi(z) = -z \frac{z+2}{2z+3}$.*

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(1) \Rightarrow (2).

- Take $y \in K$. Solve the equation $\phi(u) = y$ with $u \in K$.
If $y = \infty$, then we may take $u = -3/2$

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- Since f_u is equivalent to a rational function with K -coefficients, the proposition implies that $u \in K$.



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- By the lemma, f is equivalent to f_u for some $u \in \bar{K}$.
- The remaining critical point is $\phi(u)$. By assumption, both solutions of $\phi(z) = \phi(u)$ lie in $\mathbb{P}^1(K)$, so that $u \in K$. That is, f is equivalent to a rational function with K -coefficients.



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Corollary (Faber, T.)

If $f \in \mathbb{C}(z)$ is a cubic rational function with only real critical points, then f is equivalent to a real rational function.

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- $\phi(-3/2) = \infty$, and if $y \in \mathbb{R}$, then the equation $\phi(z) = y$ is equivalent to a quadratic equation with discriminant $4(y^2 - y + 1) = (2y - 1)^2 + 3 > 0$.

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- Hence $\phi(z) = y$ has a real solution.



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That is can we show there are other fields K which are ϕ -perfect, where $\phi(z) = -z \frac{z+2}{2z+3}$?

- Number fields are not ϕ -perfect for any ϕ with $\deg(\phi) \geq 2$. We can show this using a canonical height argument.
- If $\phi(z) = -z \frac{z+2}{2z+3}$, the field \mathbb{Q}_p is not ϕ -perfect.

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Question:

Fix a field K . Under what conditions must a rational function defined over \overline{K} with only K -rational critical points be equivalent to a rational function defined over K ?

Eberhart finds such conditions for fields of characteristic 2 and 3.

Further Thoughts

- A general rational function of degree d has $2d + 1$ free parameters (coefficients) and $2d - 2$ critical points.

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- $d = 3$: $2d - 5 = 1$. Express the remaining critical point as a function of the free parameter.
- Is it possible to solve for the critical points as explicit functions of parameters for $d > 3$?

Further Thoughts continued

- Bézout's Theorem gives an upper bound of 2^{2d-5} solutions for a general system of $2d - 5$ conics, while Goldberg bounds the number of distinct solutions by the smaller quantity

$$\frac{1}{d} \binom{2d-2}{d-1} \approx \frac{8}{\sqrt{\pi} d^{3/2}} 2^{2d-5}.$$

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- This suggests a substantial amount of extra structure in our system of equations, which may make progress possible.

Quartic

Example

$$f(z) = \frac{(z^4 + az^3 + bz^2)}{cz^2 + dz + 1 + a + b - c - d}$$

where $d = \frac{(3a^2 + 5ab + 2b^2 - 2ac - 2bc + 7a + 6b - 2c + 4)}{a + b + 1}$.

The critical points are:

$$t_1 = -(ac + 9a + 6b - 4c + 12)/(2c)$$

$$t_2 = (6a^2 + 4ab - 3ac + 9a + 2b)/(2c)$$

$$t_3 = -b(2a + b - c + 3)/c$$

THANK YOU!