Determining a Number Through its Sum of Divisors

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Introduction of σ_k

- For $n = p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n}$ we define

$$\sigma_k(n) = \sum_{\substack{d|n \\ d>0}} d^k = \prod_{i=1}^{n} \frac{p_i^{k(e_i+1)} - 1}{p_i^k - 1}$$

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- Note that $\sigma_0 : \mathbb{N} - \{0\} \rightarrow \mathbb{N}$ is the number of divisors function and σ_1 is the sum of divisors function.
- Question: When can we determine a number based off of its σ_k value?

When is σ_k injective?

1. Input an upper bound n and a k value;
2. compute $\sigma_k(a)$ for all n in $2 \leq a \leq n$;
3. remove duplicates from the list of $\sigma_k(a)$ outputs;
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- Computations show that σ_4's interval of injectivity is $2 \leq n \leq 60,000,000$.

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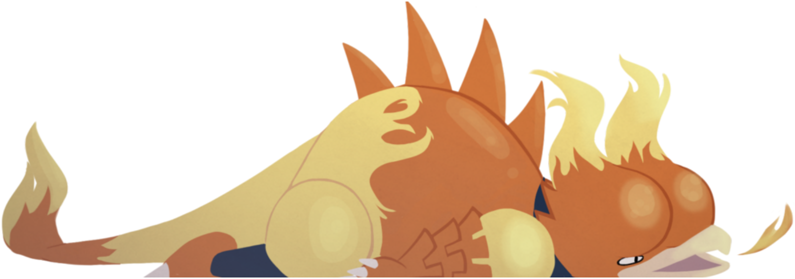

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Is $\psi_2(n)$ injective?

Some Observations

Note that when $n = p_1^{e_1} p_2^{e_2} \dots p_n^{e_n}$,

$$\sigma_0(n) = (e_1 + 1)(e_2 + 1) \dots (e_n + 1).$$

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- $\psi_2(n)$ determines n when $\sigma_0(n) = 3$. Why?
  - $n = p^2$ and $\sigma_1(n) = 1 + p + p^2$.
- When $\sigma_0(n) = q$ for a prime q, $\sigma_1(n)$ determines n since $n = p^{q-1}$ for some prime p.

The case when $\sigma_0(n) = 4$

Note that if $\sigma_0(n) = 4$ then $n = p^3$ or qr for primes p, q, r. If $\psi_2(n)$ were to fail to be injective, two cases would arise.
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$$\sigma_1(m) = 1 + p + p^2 + p^3 = 1 + q + r + qr = \sigma_1(n) \tag{1}$$

$$\sigma_2(m) = 1 + p^2 + p^4 + p^6 = 1 + q^2 + r^2 + q^2r^2 = \sigma_2(n) \tag{2}$$

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Mod q:

$$\begin{aligned}
2p^3 + 2p^4 + 2p^5 + p^2 + p^4 + p^6 &= r^2 \\
r^2 + 2p^3 + 2p^4 + 2p^5 &= r^2 \\
2p^3 + 2p^4 + 2p^5 &= 0 \\
2p^2(p + p^2 + p^3) &= 0 \\
2p^2 r &= 0.
\end{aligned}$$

What about when $\sigma_0(n) = 8$?

- Such a number is of the form p^7, pqr, or p^3q for primes p, q, and r.
- That would give us five cases to check.
  - $n = p_1^7$, $m = p_2p_3p_4$;
  - $n = p_1^7$, $m = p_2^3p_3$;
  - $n = p_1p_2p_3$, $m = p_4^3p_5$;
  - $n = p_1p_2p_3$, $m = p_4p_5p_6$;
  - $n = p_1^3p_2$, $m = p_3^3p_4$.

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  - $n = p_1p_2p_3$, $m = p_4p_5p_6$;
  - $n = p_1^3p_2$, $m = p_3^3p_4$.
- If we were to use the same method on just one case then we would be forced to compute

$$(p_1 + p_1^2 + p_1^3 + p_1^4 + p_1^5 + p_1^6 + p_1^7)^2$$

as well as

$$(p_2 + p_3 + p_4 + p_2p_3 + p_2p_4 + p_3p_4 + p_2p_3p_4)^2.$$

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- When $\sigma_0(n) = 8$, ψ_2's interval of injectivity is at least $2 \leq n \leq 10^{16}$.
- When $\sigma_0(n) = 9$, ψ_2's interval of injectivity is at least $2 \leq n \leq 10^{37}$.
- When $\sigma_0(n) = 10$, ψ_2's interval of injectivity is at least $2 \leq n \leq 10^{42}$.

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$$\psi_2(48142241) = \psi_2(48374911) = (32,\ 59609088,\ 2369077236500000)$$

Further Questions

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- Can we generalize σ_k for larger integer rings? (Spira)
- Geometric approach using intersection of affine varieties.
- What happens when we let k vary over all real numbers?
- Is ψ_k ever injective?

THANK YOU!

Sources

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