Solving families of simultaneous Pell equations

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Simultaneous *Pellian* **Equations**

$$aX^2 - bY^2 = c, \; dY^2 - eZ^2 = f$$

Ljunggren, Baker and Davenport, Grinstead, Bennett, Anglin, Cipu and Mignotte, Okazaki, Yuan, and many others.

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For $n \geq 1$, define $\alpha^n = t_n + u_n\sqrt{a}$ and $\beta^n = w_n + v_n\sqrt{b}$, then

$$u_n = \frac{\alpha^n - \alpha^{-n}}{2\sqrt{a}}, \quad v_n = \frac{\beta^n - \beta^{-n}}{2\sqrt{b}}.$$

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A solution to the above system of equations is the same as a solution to

$$u_n = \frac{\alpha^n - \alpha^{-n}}{2\sqrt{a}} = \frac{\beta^m - \beta^{-m}}{2\sqrt{b}} = v_m.$$

A Theorem of M.A. Bennett

Theorem (Bennett, 1998) For any pair of distinct positive integers a and b, there are at most **three** positive integer solutions (x, y, z) to the system of equations

$$x^2 - ay^2 = 1,\ z^2 - by^2 = 1.$$

An Improvement by Ping Zhi Yuan

Theorem (Yuan, 2006) If a and b are distinct positive integers satisfying $\max\{a, b\} > 1.4 \cdot 10^{57}$, there are at most **two** positive integer solutions (x, y, z) to the system of equations

$$x^2 - ay^2 = 1,\ z^2 - by^2 = 1.$$

The State of the Art

Theorem (Bennett, Cipu, Mignotte, Okazaki 2006) For any pair of distinct positive integers a and b, there are at most **two** positive integer solutions (x, y, z) to the system of equations

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- Baker-Davenport Lemma (LLL).
- Certain elementary arguments.
- Computations.

A Variant

Theorem (Cipu, Mignotte)

The system of Pell equations

$$x^2 - ay^2 = 1, y^2 - bz^2 = 1$$

has at most **two** solutions in positive integers x, y, z.

Yuan's Improvement

Theorem (Yuan)

For all $a \geq 1$, the system of Pell equations

$$x^2 - (4a^2 - 1)y^2 = 1, y^2 - bz^2 = 1$$

has at most **one** solution in positive integers x, y, z.

Note: Although this is best possible, this general upper bound is rarely attained.

Solving Subfamilies

Theorem (Jian Hua Chen, 2015)

For any prime p, the system of Pell equations

$$x^2 - 24y^2 = 1, y^2 - pz^2 = 1$$

has **no** solutions in positive integers x, y, z, except only for

$$p = 2 \text{ and } p = 11,$$

in which case the only positive integer solutions are $(x, y, z) = (485, 99, 70)$ and $(49, 10, 3)$ respectively.

Note: this is a special case of

$$x^2 - (m^2 - 1)y^2 = 1, \;\; y^2 - pz^2 = 1.$$

Work in Progress with Paul Voutier

Theorem

Let p denote a rational prime.
Let $5 < m < 10^6$ be a positive integer for which

- $m \equiv 5 \pmod 8$ and
- $(m-1)/4$ is a prime power.

Then the system of equations

$$x^2 - (m^2-1)y^2 = 1, \;\; y^2 - pz^2 = 1$$

has **no** solutions in positive integers x, y, z, except only for

$$(m, p) \in \{(13, 3), (29, 2)\},$$

in which for each case there is the only one solution:

$(x, y, x) = (337, 26, 15)$ and $(97469, 3363, 2378)$

respectively.

Ingredients of the Proof

• Factorization Identities of integers in Lucas Sequences.

• An old Theorem on $b^2X^4 - DY^2 = 1$ by Bennett-W.

• A Theorem on Primitive Divisors by Cam Stewart and Bilu-Hanrot-Voutier.

• The heart of the proof lies in solving the family of Thue equations

$$x^4 - (2k^2 + 2k)x^2y^2 + k^4y^4 = 1,$$

where $k = (m - 1)/8$.

Conjecture No nontrivial solutions (i.e. with $Y \neq 0$) for all $k > 3$, and the Theorem holds for all $m \equiv 5 \pmod 8$.

Saving Cycles

Rewrite $x^4 - (2k^2 + 2k)x^2y^2 + k^4y^4 = 1$ as

$$X^2 - (2k+1)k^2Y^4 = 1,$$

and use the following refinement Ljunggren's Theorem (Togbe, Voutier, W):

Theorem Let $D > 1$ denote a nonsquare integer, $\mu_D = T + U\sqrt{D}$ denote the minimal unit with norm 1 in $\mathbb{Z}[\sqrt{D}$, and

$$\mu_D^i = T_i + U_i\sqrt{D}. \ (i \geq 1)$$

If there are two indices i, j for which U_i, U_j are squares, then

$$(i, j) = (1, 2) \text{ or } (1, 4).$$

If there is only one such i, then either $i = 1, 2$ or p where p is a prime of the form $4t + 3$, in which case U_1 is of the form pv^2.

The **upshot** is, Chen's Theorem (JNT, 2015) on

$$x^2 - 24y^2 = 1, y^2 - pz^2 = 1$$

can be reduced to noticing that the only positive integer solutions (X, Y) to

$$X^2 - 3Y^4 = 1$$

arise from $\mu_3 = 2 + \sqrt{3}$ and its square

(i.e. $(X, Y) = (2, 1), (7, 2)$),