Genus numbers of Eisenstein polynomials

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Eisenstein polynomials

Definition

Recall that

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$$

is called an Eisenstein polynomial if for some prime p we have:

1. $p | a_i$ *for* $i = 0, 1, \ldots, d-1$
2. $p^2 \nmid a_0$
3. $p \nmid a_d$

For this talk, we only consider monic polynomials, where $a_d = 1$.

Example

$x^3 + 105x^2 + 315x + 210$ *is Eisenstein at* $p = 3, 5, 7$.

The genus number

The following is motivated by the study of the genus field of an algebraic number field.

Definition

Suppose f is Eisenstein at the primes in $P = \{p_1, \ldots, p_\ell\}$ and no other primes. Let e be the number of p in P such that $p \equiv 1 \pmod{d}$, and add $+1$ to e in the case where $d \in P$ and

$$a_1 \equiv \cdots \equiv a_{d-2} \equiv a_0 + a_{d-1} \equiv 0 \pmod{d^2}.$$

We define the genus number of $f(x)$ to be $g_f = d^e$.

Example

When $f = x^3 + 105x^2 + 315x + 210$, we have $g_f = 9$.

Main Question

Fix an odd prime degree d.

Question

What proportion of Eisenstein polynomials of degree d have genus number one?

Remark

For this question to make sense, we must choose an ordering. We will order out polynomials by height.

Definition

The height of a polynomial f is defined to be $max\{|a_0|, ..., |a_{d-1}|\}$.

Main Question

Definition

Let $\mathcal{E}_d(H)$ denote the collection of all monic Eisenstein polynomials of height at most H. Also, let $\mathcal{E}_d^\star(H) = \{f \in \mathcal{E}_d(H) : g_f = 1\}$.

H	10	20	30	40	50
$\frac{\#\mathcal{E}_d^\star(H)}{\#\mathcal{E}_d(H)}$	0.9581	0.9537	0.9377	0.9400	0.9419

Question

Does $\lim_{H\to\infty} \frac{\#\mathcal{E}_d^\star(H)}{\#\mathcal{E}_d(H)}$ *exist and what is its value?*

Counting Eisenstein polynomials

Theorem (Dubickas 2003, Heyman–Shparlinski 2013)

$$\#\mathcal{E}_d(H) = \theta_d (2H)^d + \begin{cases} O(H^{d-1}) \text{ if } d > 2 \\ O(H(\log H)^2) \text{ if } d = 2 \end{cases}$$

where

$$\theta_d = 1 - \prod_p \left(1 - \frac{p-1}{p^{d+1}}\right).$$

Idea of Proof

The inclusion-exclusion principle gives

$$\#\mathcal{E}_d(H) = -\sum_{s=2}^{H} \mu(s)\#\mathcal{G}_d(s, H),$$

where $\#\mathcal{G}_d(s, H)$ is the set of monic polynomials of height at most H satisfying $s|a_i$ for $i = 0, \ldots, d-1$ and $gcd(\frac{a_0}{s}, s) = 1$.

One can show

$$\#\mathcal{G}_d(s, H) = \frac{2^d H^d \varphi(s)}{s^{d+1}} + O\left(\frac{H^{d-1} 2^{\omega(s)}}{s^{d-1}}\right).$$

Main Result

Theorem (C.–M.)

$$\#\mathcal{E}_d^\star(H) = \theta_d^\star(2H)^d + \begin{cases} O(H^{d-1}) \text{ if } d > 2 \\ O(H(\log H)^2) \text{ if } d = 2 \end{cases}$$

where

$$\theta_d^\star = 1 - \frac{d-1}{d^{2d}} - \left(1 - \frac{(d-1)(d^{d-1}+1)}{d^{2d}}\right) \prod_{p \neq d, p \not\equiv 1 (d)} \left(1 - \frac{p-1}{p^{d+1}}\right)$$

Corollary

$$\lim_{H \to \infty} \frac{\#\mathcal{E}_d^\star(H)}{\#\mathcal{E}_d(H)} = \frac{\theta_d^\star}{\theta_d}$$

For example, $\frac{\theta_3^\star}{\theta_3} \approx 0.9681$.

Moreover,

$$\lim_{d \to \infty} \frac{\theta_d^\star}{\theta_d} = 1$$

Thank you

Thanks for your attention!