Coprime Mappings on n-Sets

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Coprime mappings

A bijection $f : A \to B$ on two sets of integers A and B is a *coprime mapping* if $\gcd(a, f(a)) = 1$ for all $a \in A$.

Daykin & Baines (1963): Investigated the existence of coprime mappings on sets of consecutive integers.

Main Result: If

$$A = \{1, 2, \dots, n\} \text{ and } B = \{n+1, n+2, \dots, 2n\}$$

then a coprime mapping from A onto B always exists.

Proof of D.J. Newman's Coprime Mapping Conjecture

Pomerance & Selfridge (1980): If n is a positive integer and B is any set of n consecutive integers, then there is a coprime mapping $f : \{1, 2, 3, \dots, n\} \to B$.

LR & Small (2009): Generalized to sets in arithmetic progression. Let $S = \{a + tb \,|\, 0 \leq t \leq n - 1\}$. Then there is a coprime mapping $g : \{1, 2, \dots, n\} \to S$ if and only if every common prime divisor of a and b is greater than n.

We refer to a set of n consecutive integers as an *n-set* and consider coprime mappings on adjacent n-sets A and B where $1 \notin A$. Take:

$$A = \{s, s+1, \ldots, s+n-1\} \text{ and } B = \{s+n, s+n+1, \ldots, s+2n-1\}.$$

The existence of a coprime mapping is not guaranteed:

- $A = \{2, 3, 4\}$ and $B = \{5, 6, 7\}$
- $n = 3$, 2 divides s, 3 divides $s + 1$
- $n = 10$, $3 \cdot 5 \cdot 7 \cdot 11 = 1155$, $A = \{1143, 1144, 1145, 1146, 1147, 1148, 1149, 1150, 1151, 1152\}$, $B = \{1153, 1154, 1155, 1156, \ldots, 1162\}$
- $A = \{9, 10, 11, 12\}$ and $B = \{13, 14, 15, 16\}$
- $A = \{66, 67, 68, 69, 70, 71\}$ and $B = \{72, 73, 74, 75, 76, 77\}$

The smallest adjacent n-sets A and B without a coprime mapping

n	Smallest Element of A	Smallest Element of B
2	3	5
3	2	5
4	9	13
5	9	14
6	66	72
7	65	72
8	50	58
9	51	60
10	1143	1153
11	1143	1154
12	14999	15011
13	14999	15012
14	14999	15013
15	14999	15014
16	255237	255253
17	255237	255254

Conjecture: Let n be a positive integer, $n \neq 3$, and A and B be adjacent n-sets with $n \in A$. Then a coprime mapping $f : A \to B$ exists.

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We prove the conjecture holds in two special cases:

1. When $A = \{2, 3, 4, \ldots, n+1\}$, and
2. When n or $n+1$ is prime.

We computationally verify that the conjecture holds for $n \leq 600$.

Special case: $A = \{2, 3, \ldots, n+1\}$

Theorem

Let $n > 3$. If $A = \{2, 3, \ldots, n+1\}$ and $B = \{n+2, \ldots, 2n+1\}$, then there exists a coprime mapping $f : A \to B$.

Outline of proof:

- Map the evens in A to the odds in B by using the result of Daykin & Baines / Pomerance & Selfridge.
- Map odds in A to the evens in B by dividing the evens in B by powers of two to reduce the problem to finding a coprime mapping from the set of odds in A to itself. Then use the lemma.

Lemma

Let $A = \{s + bt \mid 0 \leq t \leq n-1\}$ be a set of n integers in arithmetic progression. Then there exists a coprime mapping $f : A \to A$ if and only if $\gcd(s, b) = 1$ and s is odd if n is odd.

Example: $n = 10$

$A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
$B = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

Separate into odds/evens:

$A_{evens} = \{2, 4, 6, 8, 10\}$ $\quad$ $A_{odds} = \{3, 5, 7, 9, 11\}$
$B_{odds} = \{13, 15, 17, 19, 21\}$ $\quad$ $B_{evens} = \{12, 14, 16, 18, 20\}$

Coprime mapping on evens in A:

Use the coprime mapping $\{1, 2, \ldots, 11\} \to \{12, 13, \ldots, 22\}$:

$2 \to 13, 4 \to 15, 6 \to 17, 8 \to 19, 10 \to 21$

Coprime mapping on odds in A:

Divide the evens in B by powers of 2: $B^*_{evens} = \{3, 7, 1, 9, 5\}$. Use the lemma to construct a coprime mapping $A_{odds} \to B^*_{evens}$. Return powers of 2 to get coprime mapping $A_{odds} \to B_{evens}$.

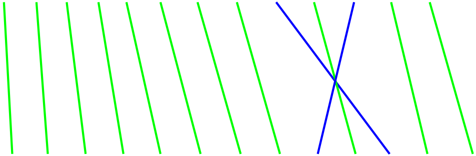
Special case: n or $n+1$ is prime

Theorem

Let $p > 3$ be prime, and A and B be adjacent p-sets with $p \in A$. Then a coprime mapping $f : A \to B$ exists.

Example: $n = 13$, $a_0 = 5$

$A = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17\}$



$B = \{18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$

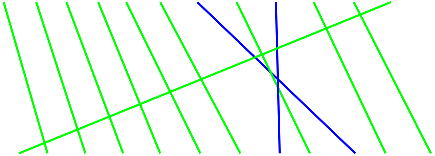
(5, 18), (6,19), (7,20), (8,21), (9,22), (10,23), (11,24), (12,25), **(13,28)**, (14,27), **(15,26)**, (16,29), (17,30)

Theorem

Let $n > 3$ be a positive integer such that $n+1$ is prime. Let A and B be adjacent n-sets with $n \in A$. Then a coprime mapping $f : A \to B$ exists.

Example: $n = 12$, $s = 5$

$A = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$



$B = \{17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28\}$

(5, 18), (6,19), (7,20), (8,21), (9,22), (10,23), **(11,26)**, (12,25), **(13,24)**, (14,27), (15,28), (16,17)

Algorithm for a coprime mapping $f : A \to B$

$$A = \{s, s+1, \ldots, s+n-1\}$$
$$B = \{s+n, s+n+1, \ldots, s+2n-1\}$$

Pomerance and Selfridge prove D. J. Newman's coprime mapping conjecture ($s = 1$) by providing an algorithm for the construction of the desired coprime mapping, and proving that their algorithm is always successful. They also briefly describe a simpler algorithm.

It is the main idea behind their simpler algorithm that we use to computationally verify our conjecture for $n \leq 600$.

General idea: Let ϕ denote Euler's function and relabel the integers in A as $a_1, a_2, \ldots, a_n$ where $\phi(a_i)/a_i \leq \phi(a_{i+1})/a_{i+1}$ for $1 \leq i < n$. Inductively define $f(a_i)$ as the least integer in B coprime to a_i and not equal to $f(a_1), \ldots, f(a_{i-1})$.

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Rough description of the algorithm:

1. Input n.
2. Let $s := 3$.
3. Let A and B be adjacent n-sets such that the smallest element of A is s.
4. Order A_{evens}, A_{odds}, B_{evens}, B_{odds} by increasing values of $\phi(k)/k$.
5. Construct a coprime mapping from A_{evens} onto B_{odds} by inductively mapping each element of A_{evens} to the first element in B_{odds} that it is coprime to.
6. Construct a coprime mapping from A_{odds} onto B_{evens} in the same way.
7. If a pair of elements is left over, run back through the pairs already matched until a pair is found that can be swapped to give two coprime pairings.
8. Repeat Steps 3–7 for $s = 4, 5, \ldots, n$.

Example of algorithm: $n = 11$, $s = 6$

$A = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$
$B = \{17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27\}$

Split into evens/odds and sort by $\phi(k)/k$:

$A_{evens} = \{6, 12, 10, 14, 8, 16\}$ $A_{odds} = \{15, 9, 7, 11, 13\}$
$B_{odds} = \{21, 27, 25, 17, 19, 23\}$ $B_{evens} = \{18, 24, 20, 22, 26\}$

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Use algorithm to find a coprime mapping from A_{evens} onto B_{odds}:
$6 \to 25$

Example of algorithm: $n = 11$, $s = 6$

$A = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$
$B = \{17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27\}$

Split into evens/odds and sort by $\phi(k)/k$:
$A_{evens} = \{6, 12, 10, 14, 8, 16\}$ $A_{odds} = \{15, 9, 7, 11, 13\}$
$B_{odds} = \{21, 27, 25, 17, 19, 23\}$ $B_{evens} = \{18, 24, 20, 22, 26\}$

Use algorithm to find a coprime mapping from A_{evens} onto B_{odds}:
$6 \to 25$, $12 \to 17$, $10 \to 21$, $14 \to 27$, $8 \to 19$, $16 \to 23$

Example of algorithm: $n = 11$, $s = 6$

$A = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$
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Use algorithm to find a coprime mapping from A_{evens} onto B_{odds}:
$6 \to 25$, $12 \to 17$, $10 \to 21$, $14 \to 27$, $8 \to 19$, $16 \to 23$

Use algorithm to find a coprime mapping from A_{odds} onto B_{evens}:
$15 \to 22$, $9 \to 20$, $7 \to 18$, $11 \to 24$, $13 \to 26$

Example of algorithm: $n = 11$, $s = 6$

$A = \{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$
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Split into evens/odds and sort by $\phi(k)/k$:
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Use algorithm to find a coprime mapping from A_{evens} onto B_{odds}:
$6 \rightarrow 25$, $12 \rightarrow 17$, $10 \rightarrow 21$, $14 \rightarrow 27$, $8 \rightarrow 19$, $16 \rightarrow 23$

Use algorithm to find a coprime mapping from A_{odds} onto B_{evens}:
$15 \rightarrow 22$, $9 \rightarrow 20$, $7 \rightarrow 18$, $11 \rightarrow 24$, $13 \rightarrow 26$

Swap to complete the mapping:
$15 \rightarrow 22$, $9 \rightarrow 20$, $7 \rightarrow 18$, $11 \rightarrow 26, 13 \rightarrow 24$

Computational Result

For $n \leq 600$ we successfully implemented our algorithm in SageMath. This verifies:

Let $4 \leq n \leq 600$ *and* A *and* B *be adjacent* n*-sets with* $n \in A$*. Then a coprime mapping* $f : A \to B$ *exists.*

Application to prime trees

Definition

A graph with vertex set V is said to be *prime* if its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge xy the labels assigned to x and y are coprime.

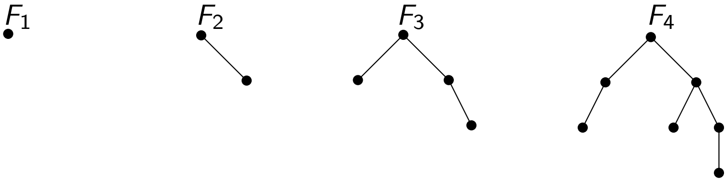
Around 1980, Entringer conjectured that all trees are prime. Little progress was made on this conjecture until 2011 when Haxell, Pikhurko, Taraz proved that all large trees are prime. Various classes of trees are known to be prime.

We wanted to prove that all Fibonacci trees are prime.

Fibonacci trees

Definition

The *Fibonacci tree* F_n is defined as the binary tree created by adjoining to a solitary vertex the Fibonacci tree F_{n-2} as a left subtree and the Fibonacci tree F_{n-1} as a right subtree. By definition, F_0 is the empty tree, and F_1 is the trivial tree consisting of a single vertex.



Let f_n denote the nth Fibonacci number $(1, 1, 2, 3, 5, 8, 13, \ldots)$.

- Number of leaves of the tree F_n is f_n.
- Number of vertices of F_n is $f_{n+2} - 1$.

Coprime mappings on Fibonacci n-sets

Theorem

Let $N \geq 5$. If there exists a coprime mapping between the f_n-sets $A_n = \{f_{n-1}, \ldots, f_n, \ldots, f_{n+1} - 1\}$ and $B_n = \{f_{n+1}, \ldots, f_{n+2} - 1\}$ for all $5 \leq n \leq N$, then the Fibonacci tree F_N is prime.

We used our algorithm: The first 30 Fibonacci trees are prime!

Note that F_{30} is a tree on 2,178,308 vertices. The computation took 27 hours.

Thank you