Special Values of L-functions in Drinfeld Modules

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December 17, 2016

Motivation

- 1974 Drinfeld: Shtuka correspondence between Drinfeld modules and vector bundles
- Rank 1 Drinfeld modules $\Longrightarrow$ vector bundles are actually line bundles
- Concrete shtuka function, determines much of arithmetic

Question

Can we write down explicit formulas (as in Carlitz) for arithmetic objects using the shtuka function?

Answer

Yes, three applications

Overview

1. Notation and Drinfeld modules
2. Exponential function and period
3. Reciprocal sums and L-series

Notation

- $q = p^r$, p prime (for talk assume $p > 3$)
- $\mathbb{F}_q$ field with q elements
- $E/\mathbb{F}_q$, elliptic curve given by the equation

$$E : y^2 = t^3 + at + b$$

- $\mathbf{A} = \mathbb{F}_q[t, y]$ coordinate ring of E
- $\mathbf{K} = \mathbb{F}_q(t, y)$ its fraction field

Notation

- $A = \mathbb{F}_q[\theta, \eta]$
- $K = \mathbb{F}_q(\theta, \eta)$
- Still have $\eta^2 = \theta^3 + a\theta + b$
- Important! Two copies of $\mathbb{F}_q(t, y)$
- In general: A = scalars, $\mathbf{A}$ = operators
- Map $\iota : \mathbf{A} \to A$, canonical isomorphism ($\iota(t) = \theta$)

Notation

- $\overline{K}$ algebraic closure of K
- K_∞ completion of K at infinite place
- $\mathbb{C}_\infty$ completion of $\overline{K}_\infty$
- Extend scalars of function field to $\mathbb{C}_\infty(t, y)$
- $\Xi = (\theta, \eta)$ is an K-rational point on E with weighted degree $(2, 3)$

Twisting

- For $a \in \mathbb{C}_\infty[t, y]$, and $a = \sum_{i,j} a_{ij} t^i y^j$, and $k \in \mathbb{Z}$ define twisting
$$a^{(k)} = \sum_{i,j} a_{ij}^{q^k} t^i y^j$$
- Ie. twisting only affects the θ, η variables
- Define twisting on $E(\mathbb{C}_\infty)$, eg. $\Xi^{(1)} = (\theta^q, \eta^q)$

Twisting via Operators

- $K\{\tau\}$, twisted polynomial ring, for $a \in K$

$$\tau a = a^q \tau$$

- $K\{\tau\}$ acts on $a \in \mathbb{C}_\infty(t, y)$

$$\tau^i a = a^{(i)}$$

and K acts by multiplication

Drinfeld Module

A Drinfeld module is an $\mathbb{F}_q$-module homomorphism $\rho : \mathbf{A} \to \overline{K}\{\tau\}$, such that

$$\rho_a = \iota(a) + a_1\tau + \cdots + a_i\tau^i,$$

which provides an $\mathbf{A}$ action on $\mathbb{C}_\infty$. The rank r of ρ is the unique integer such that $i = r \deg a$ for all $a \in \mathbf{A}$.

Example: If $\rho_t = \theta + \tau$, then

$$\rho_t(x) = \theta \cdot x + x^q.$$

Exponential Function

The exponential function, $\exp_\rho : \mathbb{C}_\infty \to \mathbb{C}_\infty$, defined as

$$\exp_\rho(z) = \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i} \in \overline{K}[[z]],$$

is the unique $\mathbb{F}_q$-linear power series satisfying

$$\exp_\rho(\iota(a)z) = \rho_a(\exp_\rho(z)), \quad a \in \mathbf{A}.$$

Drinfeld divisor

Drinfled (1974): There exists a point V on $E(\overline{K})$ such that

$$V - V^{(1)} = \Xi,$$

with coordinates denoted $V = (\alpha, \beta)$. The point V is called the Drinfeld divisor.

Shtuka function

The function in $f \in \overline{K}(t, y)$ with divisor

$$\mathrm{div}(f) = (V^{(1)}) - (V) + (\Xi) - (\infty),$$

is called the shtuka function for $\mathbf{A}$ (suitably normalized). Explicitly,

$$f(t, y) = \frac{y - \eta - m(t - \theta)}{t - \alpha} = \frac{\nu(t, y)}{\delta(t)},$$

where m is the slope between $V^{(1)}$ and Ξ.

The ω_ρ function

Application: Calculating the Period. Following Anderson and Thakur we define the function

$$\omega_\rho = \xi^{1/(q-1)} \prod_{i=0}^{\infty} \frac{\xi^{q^i}}{f^{(i)}},$$

where $\xi \in \overline{K}$ is a normalizing factor.

Essential properties:

- $(\tau - f)(\omega_\rho) = 0$
- Simple poles at $\Xi^{(i)}$ for $i \geq 0$

The Period and ω_ρ

Theorem (G., Papanikolas)

Denote $\pi_\rho := -\operatorname{Res}_\Xi(\omega_\rho \cdot \frac{dt}{2y})$. *Then* π_ρ *generates* $\ker(\exp_\rho)$ *and we have the product formula*

$$\pi_\rho = -\frac{\xi^{q/(q-1)}}{\delta^{(1)}(\Xi)} \prod_{i=1}^{\infty} \frac{\xi^{q^i}}{f^{(i)}(\Xi)},$$

where $\delta(t)$ *is the denominator of the shtuka function.*

Proof.

Anderson generating functions. □

Reciprocal Sums

Application: Reciprocal Sums. We derive new formulas for reciprocal sums

$$S_i = \sum_{a \in A_{i+}} \frac{1}{a},$$

where A_{i+} are the monic elements in A of degree i.

- Studied first by Carlitz in 1935
- Thakur in 1992
- Recently by Pellarin, Anglès, Simon and Perkins (others)

Reciprocal Sums

Theorem (G., Papanikolas)

The following formula holds for $i \geq 2$,

$$S_i = \sum_{a \in A_{i+}} \frac{1}{a} = \frac{\nu^{(i)}}{g_i^{(1)} \cdot f^{(1)} \cdots f^{(i)}} \Bigg|_{\Xi},$$

where $g_i(t,y) \in \overline{K}(t,y)$ is an easily understood linear polynomial and $\nu(t,y)$ is the numerator of the shtuka function.

We also get similar formulas for sums over elements in certain prime ideals.

Example

Let $E : y^2 = t^3 - t - 1$ over $\mathbb{F}_3$. Recall $\Xi = (\theta, \eta)$ is a K-rational point on E. Monic polynomials in A are

- Degree 0: 1
- Degree 1: $\emptyset$
- Degree 2: θ, $\theta + 1$, $\theta - 1$
- Degree 3: η, $\eta + 1$, $\eta - 1$, $\eta + \theta$, $\eta - \theta$, $\eta + \theta + 1$, $\eta + \theta - 1$, $\eta - \theta + 1$, $\eta - \theta - 1$

Example

We calculate

$$S_2 = \frac{1}{\theta} + \frac{1}{\theta+1} + \frac{1}{\theta-1} = \frac{1}{\theta-\theta^3}.$$

Also, we can write

$$f(t,y) = \frac{\nu(t,y)}{\delta(t)} = \frac{y-\eta-\eta(t-\theta)}{t-\theta-1}$$

and

$$g_i^{(1)}(t,y) = y - \eta^3 + \frac{\eta^{3^i}+\eta^3}{\theta^{3^i}-\theta^3}\cdot(t-\theta^3-1).$$

Example

One then checks (on the computer) that

$$S_2 = \frac{1}{\theta - \theta^3} = \frac{\nu^{(2)}(\Xi)}{g_2^{(1)}(\Xi) f^{(1)}(\Xi) f^{(2)}(\Xi)}.$$

Ideas Behind Proof

We use a generalization of a lemma of Simon.

Lemma

Let $s \geq 1$ *and* $i \geq 2$, *define*

$$\mathcal{T}_{i,s}(t_1, y_1, \ldots, t_s, y_s) = \sum_{a \in A_{i+}} a(t_1, y_1)a(t_2, y_2) \cdots a(t_s, y_s)$$

Then $\mathcal{T}_{i,s} = 0$ *if and only if* $s < (i-1)(q-1)$.

Ideas Behind Proof

Define a deformation of S_i

$$\mathcal{S}_i(t,y) = \sum_{a \in A_{i+}} \frac{a(t,y)}{a(\theta,\eta)}.$$

Using Simon's lemma to analyze the divisor of $\mathcal{S}_i$ we obtain the identity of functions

$$\mathcal{S}_i = S_i \cdot \frac{g_i}{\nu^{(i-1)}} \cdot f f^{(1)} \cdots f^{(i-1)}.$$

After some more work, we can specialize at Ξ and arrive at the identity.

L-series

Application: Pellarin's L-series. Define

$$L(t,y;s) = \sum_{a \in A_+} \frac{\chi(a)}{a^s} = \sum_{a \in A_+} \frac{a(t,y)}{a^s}.$$

- Compare with classical L-series where $\chi(a) = a(t,y)$ is a "quasi-character".
- Pellarin proves special values for s equals 1 in the genus 0 case (Carlitz module).
- We also consider L-series using ideal sums, a la Goss, similar to Dedekind zeta function

L-series

We are interested in the specialization at $s = 1$

$$L(t, y; 1) = \sum_{i \geq 0} \mathcal{S}_i.$$

This allows us to apply the formulas for $\mathcal{S}_i$ which we just discussed.

Theorem (G., Papanikolas)

As elements of the Tate algebra,

$$L(t, y; 1) = -\frac{\pi_\rho \, \delta^{(1)}}{f \omega_\rho}.$$

Recall:

- π_ρ = *period of exponential*
- δ = *denominator of shtuka function*
- ω_ρ = *product of reciprocal shtuka functions*

Future Directions

- (Immediate future) Examine tensor powers of $\mathbf{A}$-modules. Can we get transcendence results?
- (Less near future) Examine rank-1 modules over curves of higher genus.

Thank you for listening!