Infinite Product Exponents for Modular Forms

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Modular Forms for $\Gamma_0(N)$

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Definition

The congruence subgroup $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ is

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}.$$

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- $f(z) = (cz + d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,
- f is holomorphic at the **cusps**: the orbits of $\mathbb{P}_1(\mathbb{Q})$ under $\Gamma_0(N)$.

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- Example: The discriminant function has Fourier expansion

$$\Delta(z) = \sum_{n=0}^{\infty} \tau(n) q^n$$

where $\tau(n)$ is the Ramanujan function

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- Infinite product expansion for $f(z)$:

$$f(z) = q^h \prod_{m=1}^{\infty} (1 - q^m)^{c(m)}; \qquad c(m) \in \mathbb{C}.$$

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- $\Delta(z)$ is a modular form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$
- Infinite product expansion of $\Delta(z)$:

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

More Examples: Eta-products

- The eta-function:

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- Eta-products can be used to construct modular forms:

$$f(z) = \prod_{d|N} \eta(dz)^{r_d}.$$

More Examples: Eta-products

- Example: A weight 2 cusp form for $\Gamma_0(11)$.

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- Infinite product expansion:

$$f(z) = q\prod_{m=1}^{\infty}(1-q^m)^{c(m)}$$

$$c(m) = [2, 2, \ldots, 2, 4, 2, 2, \ldots, 2, 4, 2, 2, \ldots]$$

$c(m)$ isn't always pretty

- The weight 4 normalized Eisenstein series:

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

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A Theorem of Kohnen

Theorem (Kohnen '05)

Suppose $f(z)$ is a weight k meromorphic modular form on Γ **with no zeros or poles on** $\mathcal{H}$ *and $n \in \mathbb{Z}_{>2}$*

- *If Γ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$, then* $c(m) \ll_f \log\log m \cdot \log m$
- *If Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then* $c(m) \ll_f (\log\log m)^2$

A General Upper Bound for $c(m)$

Consider a holomorphic modular form $f(z)$ with infinite product expansion

$$f(z) = q^h \prod_{m=1}^{\infty} (1 - q^m)^{c_f(m)}.$$

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Theorem

Assume the set of roots of $f(z)$ in a fundamental domain $\mathcal{F}_N$ is $\{z_j = x_j + iy_j\}_{j=1,\ldots,r}$ with $y_1 \leq \ldots \leq y_r$ and $r \geq 1$. Then,

$$c_f(m) \ll \frac{e^{2\pi m y_r}}{m^{3/2}}.$$

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$$c_f(m) = \Omega\left(\frac{e^{2\pi m y_r}}{m^{3/2}}\right).$$

Preliminaries

- Consider $z = x + iy \in \mathcal{H}$ where $\mathcal{H} = \{z \in \mathbb{C} | y > 0\}$
- Let $q = e^{2\pi iz}$ and consider a (holomorphic) modular form $f(z)$ for $\Gamma_0(N)$ with Fourier expansion

$$f(z) = \sum_{n=h}^{\infty} a(n)q^n \qquad a(h) = 1$$

The Modular Curve

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- $z_1, z_2 \in \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ are equivalent with respect to $\Gamma_0(N)$:

$$z_1 \sim z_2 \implies \exists \sigma \in \Gamma_0(N), \quad \sigma z_1 = z_2$$

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- $\nu_z^{(N)}(f(z))$: (weighted) order of the zero of f at z on $X_0(N)$

Previous Work

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Theorem (BKO '04)

For $f(z)$ a weight k meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ and n a positive integer,

$$\sum_{d|m} c(d)d = 2k\sigma_1(m) + \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(1)} \operatorname{ord}_\tau(f) \cdot j_m(\tau).$$

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- $j_m(\tau)$: apply T_m to the j-function
- Unique modular form holomorphic on $\mathcal{H}$ beginning as

$$j_m(\tau) = q^{-m} + \sum_{n \geq 1} c_m(n) q^n$$

Motivating Work

Theorem (Choi '09)

For $f(z) = \sum_{m=h}^{\infty} a(m)q^m$ a normalized meromorphic modular form of weight k on $\Gamma_0(N)$ and $m \in \mathbb{Z}^+$,

$$\sum_{d|m} dc(d) = \sum_{z \in \mathcal{F}_N \cup \mathcal{C}_N^*} \nu_z^{(N)}(f) j_{N,m}(z) - \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy$$

$$+ \frac{24h - 2k}{N-1} N\sigma_1(m/N) + \frac{2Nk - 24h}{N-1}\sigma_1(m).$$

An Explicit Formula for $c(m)$

Proposition (Möbius Inversion)

$$c_f(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) \left(\sum_{z \in \mathcal{F}_N \cup \mathcal{C}_N^*} \nu_z^{(N)}(f) j_{N,d}(z) - \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,d}(z)) dxdy \right)$$

$$+ \begin{cases} \frac{2Nk - 24h}{N-1} & N \nmid m \\ 2k & N \mid m \end{cases}$$

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$$c_f(m) \ll \frac{e^{2\pi m y_r}}{m^{3/2}}.$$

Proof Idea

$$c_f(m) = \frac{1}{m}\sum_{d|m}\mu\left(\frac{m}{d}\right)\left(\sum_{z\in\mathcal{F}_N\cup\mathcal{C}_N^*}\nu_z^{(N)}(f)j_{N,d}(z) - \underbrace{\int_{\mathcal{F}_N}^{reg} f_\theta(z)\cdot\xi_0(j_{N,d}(z))dxdy}_{R(m)}\right) + \begin{cases}\frac{2Nk-24h}{N-1} & N\nmid m\\ 2k & N\mid m\end{cases}$$

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$$R(m) = \int_{\mathcal{F}_N}^{reg} f_\theta(z)\cdot\xi_0(j_{N,d}(z))dxdy \ll \frac{e^{2\pi m y_r}}{\sqrt{m}}$$

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$genus(X_0(N)) = 1$

$j_{N,m}(z)$ is holomorphic infinitely often

$j_{N,d}(z) \asymp \frac{e^{2\pi m y_r}}{\sqrt{m}}$

$R(m)$ vanishes infinitely often

$O(1)$ growth

$c_f(m) = \Omega\left(\frac{e^{2\pi m y_r}}{m^{3/2}}\right)$

A Stricter Bound

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Corollary

Suppose that $f(z)$ is a modular form with **no zeros or poles on the upper half plane** *and infinite product expansion*

$$f(z) = q^h \prod_{m=1}^{\infty} (1 - q^m)^{c_f(m)}$$

Then,

$$c_f(m) \ll \log m \cdot \log\log m$$

Proof Idea

- Relate $R(m)$ to weighted cusp summation:

$$f_\theta(z)_d + \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,d}(z)) dxdy = \sum_{z \in \mathcal{C}_N^*} \nu_z^{(N)}(f(z)) j_{N,d}(z)$$

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- Bound summation when $f(z)$ has no zeros or poles on $\mathcal{H}$

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- Bound summation when $f(z)$ has no zeros or poles on $\mathcal{H}$
- Upper bound using growth of $\sigma_1(m)$

Limitations on Tight Bounds

- Ω bound when the genus of $X_0(N)$ was 0 or 1 follows when cusp forms of weight 2 and level N have infinitely many vanishing Hecke eigenvalues.

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- Ω bound when the genus of $X_0(N)$ was 0 or 1 follows when cusp forms of weight 2 and level N have infinitely many vanishing Hecke eigenvalues.
- Growth of $c(m)$ is likely also exponential for higher genus modular curves
- $R(m)$ cancels with summation in cases where $c(m)$ is bounded, like η-quotients

Summary

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Summary

- Gave an exponential upper bound on the infinite product coefficients $c(m)$ of a holomorphic modular form $f(z)$
- Showed this bound on $c(m)$ was also a lower bound in the case that $f(z)$ had at least one zero in $\mathcal{H}$ and the genus of $X_0(N)$ was 0 or 1
- Found the bound to be consistent with cases where the growth of $c(m)$ was slow

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- In particular, thanks to Dr. Michael Mertens and Professor Ken Ono for their mentorship and support.