A 2-adic view of Lehmer's interesting series

Paul Thomas Young

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- "This paper marks the sixtieth anniversary of the publication of the author's first paper in this MONTHLY. The author hopes to submit manuscripts of other papers from time to time as occasion arises."
- He defined a series to be *interesting* in case there is a simple explicit formula for its nth term and at the same time its sum can be expressed in terms of known constants.

INTERESTING SERIES INVOLVING THE CENTRAL BINOMIAL COEFFICIENT

D. H. LEHMER
Department of Mathematics, University of California, Berkeley, CA 94720

The adjective "interesting" is used here in a technical sense explained as follows. A series is called interesting in case there is a simple explicit formula for its nth term and at the same time its sum can be expressed in terms of known constants. Thus

$$1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}+\cdots=2,$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n+1}}{n}+\cdots=\log 2,$$

$$1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\cdots+\frac{1}{n^4}+\cdots=\frac{\pi^4}{90},$$

are familiar examples of interesting series.

The series we plan to discuss are of two types:

$$\text{I.}\quad \sum_{n=0}^{\infty} a_n\binom{2n}{n} \qquad \text{and} \qquad \text{II.}\quad \sum_{n=0}^{\infty}\frac{a_n}{\binom{2n}{n}},$$

where the a_n are very simple functions of n. We begin with series of Type I.

Two types of interesting series

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- The series of type II are more mysterious and less well understood. The paper was apparently motivated by Apéry's 1978 proof of the irrationality of $\zeta(3)$, and van der Poorten's analysis thereof, which featured the series

$$\sum_{n=0}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \frac{1}{3}\zeta(2) \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}} = \frac{2}{5}\zeta(3).$$

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- Although it is also known that

$$\sum_{n=0}^{\infty} \frac{1}{n^4 \binom{2n}{n}} = \frac{17\pi^4}{3240},$$

Lehmer remarks that there are no known interesting series of the form

$$\sum_{n=0}^{\infty} \frac{1}{n^k \binom{2n}{n}}$$

for $k > 4$.

Interesting series of type II

The main tool used to analyze series of type II is the identity

$$\frac{2x \arcsin x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}}$$

which is valid for $|x| < 1$.

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- Corresponding to $x = 1/2$ he gives

$$\sum_{n=1}^{\infty} \frac{n}{\binom{2n}{n}} = \frac{2}{27}(\pi\sqrt{3} + 9)$$

$$\sum_{n=1}^{\infty} \frac{n^2}{\binom{2n}{n}} = \frac{2}{81}(5\pi\sqrt{3} + 54)$$

$$\sum_{n=1}^{\infty} \frac{n^3}{\binom{2n}{n}} = \frac{2}{243}(37\pi\sqrt{3} + 405)$$

The most interesting series of type II

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$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} = \frac{\pi}{2} + 1$$

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$$\sum_{n=1}^{\infty} \frac{n^2 2^n}{\binom{2n}{n}} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{n^3 2^n}{\binom{2n}{n}} = \frac{7\pi}{2} + 11$$

$$\sum_{n=1}^{\infty} \frac{n^4 2^n}{\binom{2n}{n}} = 113\pi + 355$$

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- In general,

$$\sum_{n=1}^{\infty} \frac{n^k 2^n}{\binom{2n}{n}} = a\pi + b$$

where b/a is a close approximation to π.

$$\sum \frac{2^m}{\binom{2m}{m}} = \frac{\pi}{2} + 1$$

$$\sum \frac{m 2^m}{\binom{2m}{m}} = \pi + 3$$

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In 2013 Dyson, Frankel, and Glasser published "Lehmer's interesting series" in AMM, dealing with the series

$$S_k(z) := \sum_{n=1}^{\infty} \frac{n^k z^n}{\binom{2n}{n}}$$

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- Proved Lehmer's assertion that $S_k(2) = a_k\pi + b_k$ with $a_k, b_k \in \mathbb{Q}$ and

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- Note that both sequences $\{b_k\}$ and $\{2a_k\}$ are actually sequences of *positive integers*; in fact you can find them on OEIS as sequences A180875 and A014307.
- Gave an detailed analysis of the rate of convergence.
- Also included a nice tribute to D.H.Lehmer in the epilogue, along with a tribute to π.

Lehmer's interesting series

Freeman J. Dyson

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Norman E. Frankel

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M. Lawrence Glasser

Department of Physics, Clarkson University, Potsdam, New York, USA[‡]

(Dated: March 29, 2011)

Abstract

The series

$$S_k(z) = \sum_{m=1}^{\infty} \frac{m^k z^m}{\binom{2m}{m}}$$

is evaluated in a non-recursive and closed process and it can be analytically continued beyond its domain of convergence $0 \leq |z| < 4$ for $k = 0, 1, 2, \cdots$. From this we provide a firm basis for Lehmer's observation that π emerges from the limiting behavior of $S_k(2)$ as $k \to \infty$.

VI. EPILOGUE

D. H. Lehmer was an eminent mathematician [25]. His many and varied works dealt especially with matters of numbers, the queen of mathematics, for which he had an abiding and prodigious affection and talent. Many of his elegant works bear his name in the literature. Particularly relevant here are his Machin-like " On Arccotangent Relations for pi" [26] and " A Cotangent Analogue of Continued Fractions" [27], in which he showed that every positive irrational number has a unique infinite continued cotangent representation.

It would be remiss to close our essay without a salute to π, the most renown of all constants in mathematics. Of the literally plethora of fine possible choices, we have chosen to go with,

" The value of π has engaged the attention of many mathematicians and calculators from the time of Archimedes to the present day, and has been computed from so many different formulae, that a complete account of its calculation would almost amount to a history of mathematics. "

[J. W. L. Glaisher, Messenger of Math., 25-30 (1872)]

and

" And he made a molten sea, ten cubits from the one brim to the other; it was round all about ... and a line of thirty cubits did compass it round about."

[1 Kings 7:23]

[*On The Rabbinical Approximation of Pi* [28]]

Polylogarithmic zeta functions

For $\Re(a) > 0$ and $|r| \geq 1$ we define the zeta function

$$Z_{r,k}(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathrm{Li}_k\left(\frac{1-e^{-t}}{r}\right) \frac{re^{-at}}{1-e^{-t}}\, dt$$

for $\Re(s) > 0$, where

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denotes the order k polylogarithm function.

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- If $|r| = 1$ the integral converges when $\Re(k) > 0$ and $\Re(a) + \Re(k) > 1$, and if $|r| > 1$ it is convergent for all k; in either case $Z_{r,k}(s,a)$ may be analytically continued to all $s \in \mathbb{C}$.

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- The change of variable $u = (1-e^{-t})/r$ shows that the value $Z_{r,k}(s,a)$ is a real period in the sense of Kontsevich and Zagier when $k, s \in \mathbb{Z}$ and $r, a \in \bar{\mathbb{Q}} \cap \mathbb{R}$.

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- We have the Riemann zeta function $Z_{1,1}(s,1) = s\zeta(s+1)$, and

  $$Z_{1,1}(s,a) = s\zeta(s+1,a)$$

  where $\zeta(s,a) = \sum_{m \geq 0}(m+a)^{-s}$ is the Hurwitz zeta function.

Polylogarithmic zeta functions - special cases

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$$Z_{r,0}(s,a) = \frac{r}{r-1}\Phi\left(\frac{-1}{r-1}, s, a\right)$$

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- When $r = 1$ we get *Arakawa-Kaneko zeta functions* $Z_{1,k}(s,a) = \xi_k(s,a)$, which satisfy

$$\xi_{k-1}(m,1) = \zeta^{\star}(k, \underbrace{1,\ldots,1}_{m-1}) := \sum_{n_1\geq n_2\geq\cdots\geq n_m\geq 1}\frac{1}{n_1^k n_2\cdots n_m}$$

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  in terms of *non-strict multiple zeta values* (also called *multiple zeta-star values*), whose arithmetic has been extensively studied;
- and when $r=2$ we get *alternating Arakawa-Kaneko zeta functions* $Z_{2,k}(s,a) = 2\xi_k^\star(s,a)$.

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$$Z_{r,k}(n+1, a) = \sum_{m=0}^{\infty} \frac{m! P_n(h_m^{(1)}(a), \dots, h_m^{(n)}(a))}{r^m (m+1)^k a(a+1)\cdots(a+m)}$$

when $\Re(a) > 0$. Here $P_n(x_1, \dots, x_n)$ denotes the modified Bell polynomial defined by

$$\exp\left(\sum_{n=1}^{\infty} x_n \frac{t^n}{n}\right) = \sum_{n=0}^{\infty} P_n(x_1, \dots, x_n) t^n$$

which we evaluate at generalized harmonic numbers

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  when $\Re(a) > 0$. Here $P_n(x_1, \ldots, x_n)$ denotes the modified Bell polynomial defined by

  $$\exp\left(\sum_{n=1}^{\infty} x_n \frac{t^n}{n}\right) = \sum_{n=0}^{\infty} P_n(x_1, \ldots, x_n) t^n$$

  which we evaluate at generalized harmonic numbers

  $$h_m^{(n)}(a) = \sum_{j=0}^{m} \frac{1}{(a+j)^n}.$$

- This expansion is due to Coppo and Candelpergher. However, I showed that when $|a|_p > 1$ and $s \in \mathbb{Z}_p$, this same series also converges p-adically to a p-adic polylogarithmic zeta function $Z_{p,r,k}(s, a)$ which encodes the p-adic properties of harmonic number sums.

A 2-adic view of Lehmer's interesting series

Lehmer and Dyson et al showed that the real limit of the series

$$S_k(2) := \sum_{n=1}^{\infty} \frac{n^k 2^n}{\binom{2n}{n}} = a_k \pi + b_k \qquad \text{in} \quad \mathbb{R}$$

where $\{2a_k\}$ and $\{b_k\}$ are integer sequences and $b_k/a_k \to \pi$.

$\{2a_k\}$: $1, 2, 7, 35, 226, 1787, 16717, 180560, 2211181, 30273047, 458186752, \ldots$ (A014307)

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- **Theorem.** The sum of the series in $\mathbb{Q}_2$ is

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- Perhaps this theorem is giving a very concrete illustration of this principle?

Proof of the theorem

Begin by considering the 2-adic series $S_k(2)$ as the 2-adic polylogarithmic zeta function $Z_{2,2,-(k+1)}(1, \frac{1}{2})$. Define $G(j,k)$ by

$$G(k,j) := (-1)^j \prod_{i=0}^{j-1}(2i-1) \cdot Z_{2,2,1-k}(1, \frac{3}{2} - j)$$

A lazy table of values of $G(k,j)$ is

0	2	4	26	152
1	5	23	167	1473
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- The sequence $G(1,j) = 2, 4, 26, 152, \ldots$ is twice sequence A024199 in OEIS, whose formula is

$$G(1,j) := 2(-1)^j \prod_{i=0}^{j-1}(2i-1) \cdot \sum_{i=1}^{j} \frac{(-1)^{i-1}}{2i-1}.$$

  This may be proved by functional equation and difference formulas for the functions $Z_{2,2,k}(s,a)$. We want to prove the sequence $G(k,1) = 1, 3, 11, 55, 355, \ldots$ is $G(k,1) = b_{k-1}$ where $S_k(2) = a_k\pi + b_k$.

Proof of the theorem - continued

The values in the table $G(k,j)$ satisfy the recurrence

$$2G(k,j) = (2j-1)G(k-1,j) + G(k-1,j+1).$$

This turns out to be a 2-adic version of an *alternating* analogue of Ohno's sum formula for multiple zeta values!

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- This enables us to express the 2-adic Lehmer series $b_{k-1} = G(k,1)$ in terms of the sequence $G(1,j) = 2, 4, 26, 152, \ldots$ which is twice sequence A024199 in OEIS. From there it is just a gnarly combinatorial problem, involving weighted Stirling numbers and generalized factorials, to prove that Dyson's formula for $\{b_k\}$ is obtained from these recurrences.

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- Several other series from Lehmer's paper seem to behave in a similar way in $\mathbb{Q}_2$ or $\mathbb{Q}_3$, but I haven't given proofs of them. There are other similar interesting series that I have proved.

Beta function examples

Similar examples arising from the Dirichlet beta function $\beta_0(s) = \sum_{m\geq 0}(-1)^m(2m+1)^{-s}$ include

$$\sum_{m=0}^{\infty} \frac{2^{m-1}}{(2m+1)\binom{2m}{m}} = \begin{cases} \beta_0(1) = \pi/4 & \text{in} \quad \mathbb{R} \\ \beta_{2,0}(1) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

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- The value $\beta_0(2) = G = 0.9159655\cdots$ is known as *Catalan's constant*; it is not known whether G is irrational, but its 2-adic analogue $\beta_{2,0}(2) = G_2$ was shown to be irrational by Calegari in 2004.

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- As known to Dirichlet, you get a rational multiple of a power of π from $\zeta(2k)$ or from $\beta_0(2k+1)$. Yet the only one of the remaining (real) values whose arithmetic nature has been determined to date is Apéry's 1978 proof of the irrationality of $\zeta(3)$. We have had more success determining irrationality of their 2-adic and 3-adic analogues!

Some examples involving the $k = 1$ polylogarithmic analogue of $\beta_0(s)$ include

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{n^2\binom{2n}{n}} = \begin{cases} \beta_1(1) = \frac{\pi^2}{16} & \text{in} \quad \mathbb{R} \\ \beta_{2,1}(1) = 0 & \text{in} \quad \mathbb{Q}_2. \end{cases}$$

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- The first three real values above were given by Coppo and Candelpergher.
- These examples are part of an infinite family of series of rational numbers which converge to polylogarithmic zeta values in $\mathbb{R}$ and simultaneously converge to their 2-adic counterparts in $\mathbb{Q}_2$. Again the 2-adic side is "simpler".

Real consequences

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- A convolution relation for 2-adic Arakawa-Kaneko zeta functions led me to the ordinary convolution identity

$$\begin{aligned}\sum_{j=0}^{n} B_j(a)B_{n-j}(a) &= -4\left(\mathbb{B}_{n+1}^{(3)}(1-a)+(-1)^n\mathbb{B}_{n+1}^{(3)}(a)\right)\\ &\quad -(n+1)\left(\mathbb{B}_{n}^{(2)}(1-a)+(-1)^n\mathbb{B}_{n}^{(2)}(a)\right),\end{aligned}$$

where $B_n(a)$ denotes the Bernoulli polynomial and $\mathbb{B}_n^{(k)}(a)$ denotes the *poly-Bernoulli polynomial* of order k.

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  where $B_n(a)$ denotes the Bernoulli polynomial and $\mathbb{B}_n^{(k)}(a)$ denotes the *poly-Bernoulli polynomial* of order k.

- A similar 2-adic convolution identity reveals to us that for all positive integers n, the alternating odd harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{j=1}^{k} (2j-1)^{-n} = n(1-2^{-(n+1)})\zeta(n+1) - \sum_{k=1}^{n} \beta_0(k)\beta_0(n+1-k)$$

  is expressible in terms of Riemann zeta and Dirichlet beta values, and for $n > 1$, the odd alternating harmonic series

$$\sum_{k=1}^{\infty} (2k-1)^{-n} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j}$$

  is similarly expressible as a specific rational polynomial combination of $\log 2$, Riemann zeta and Dirichlet beta values.

Thank You!