Reduction of Dynatomic Curves

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Arithmetic dynamics: $X = \mathbb{P}^1(\mathbb{Q})$ (or similar), $f \in \mathbb{Q}(z)$.

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Conjecture (Uniform Boundedness Conjecture)

There is a constant $C(d)$ so that if f is any degree-d rational function, then $|\operatorname{PrePer}_f(\mathbb{Q})| \leq C(d)$.

Elliptic modular curves

$Y_1^{\mathrm{ell}}(n)$: parametrizes pairs (E, P): E an elliptic curve, $P \in E[n]$

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What is a dynamical analogue?

Iteration of quadratic polynomials

$f_c(x) = x^2 + c$

$f_c^n(x) = f_c(f_c(\cdots(f_c(x))\cdots))$, n times

$x \in \mathbb{C}$ is *n-periodic* if $f_c^n(x) = x$, or $f_c^n(x) - x = 0$.

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Makes sense to filter out the points that are n-periodic but not d-periodic for d a proper divisor of n.

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Definition

$\Psi_n(x, c)$ is the n^{th} dynatomic polynomial

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$c = 0$: $\Psi_n(x, 0)$ is a product of cyclotomic polynomials:

$$\Psi_n(x, 0) = \prod_{d|n} (x^{2^d} - x)^{\mu(n/d)}.$$

Compare with cyclotomic polynomials:

$$C_n(x) = \prod_{d|n} (x^d - x)^{\mu(n/d)}.$$

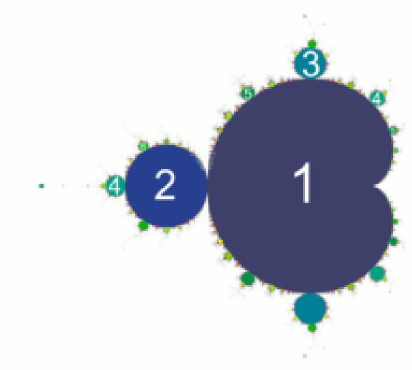
Multiple roots

Idea: solutions to $\Psi_n(x, c) = 0$ "should be" pairs (x, c) so that x has *exact* period n for $f_c(x)$.

Not quite true: there are points (x, c) where the period of x is a proper divisor of $f_c(x)$, when $f_c^n(x) - x$ has double (or higher) roots at x.

Formal period

This happens at the bifurcation points and cusps of the Mandelbrot set



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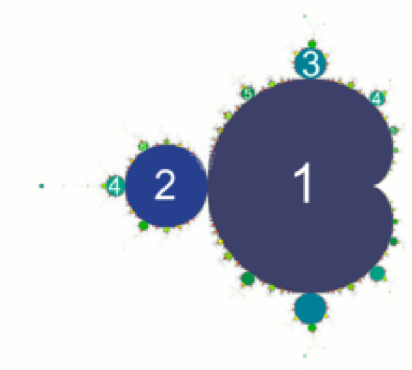

Definition

If $\Psi_n(x, c) = 0$, then we say that x has *formal period* n for $f_c(x)$.

Dynatomic curves

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Theorem (Buff, Lei)

All these dynatomic curves (X_0, X_1, Y_0, Y_1) are smooth and irreducible

Monodromy

- Branch points of $X_0(n) \to \mathbb{P}^1$ are some of the cusps of the Mandelbrot set.
- Branch points of $X_1(n) \to \mathbb{P}^1$ are some of the cusps and bifurcation points of the Mandelbrot set.

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Fact: for $X_0(n) \to \mathbb{P}^1$, monodromy around each branch point is a transposition.

Reduction

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What can we say about its reduction? Good reduction: genus of curve in characteristic p = genus of curve in characteristic 0.

Bad reduction

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Example

$X_0(5)$ has bad reduction at p iff $p = 2$ or 3701.

X_0 versus X_1

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False if n is composite: $X_0(6)$ has good reduction at 67, whereas $X_1(6)$ has bad reduction at 67.

Discriminants and related objects

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(Up to a factor of a unit)

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$\Delta_{n,n} \in \mathbb{Z}[c]$. Roots of $\Delta_{n,n}$: two orbits of formal period n collide, i.e. certain cusps of the Mandelbrot set, which are also branch points of $X_0(n) \to \mathbb{P}^1$.

Discriminant of $\Delta_{n,n}$

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If $X_0(n)$ has bad reduction at p, then $p \mid \mathrm{Disc}(\Delta_{n,n})$.

However, many primes dividing $\mathrm{Disc}(\Delta_{n,n})$ still have good reduction.

Bad reduction and Disc($\Delta_{n,n}$)

$\mathrm{Disc}(\Delta_{5,5}) = 2^{274} \cdot 3^{12} \cdot 31^{27} \cdot 3701^{1} \cdot 4217^{3}$

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If $v_p(\mathrm{Disc}(\Delta_{n,n})) = 1$, then $X_0(n)$ has bad reduction at p.

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But:

Theorem

If n is odd and $v_p(\mathrm{Disc}(\Delta_{n,n})) = 1$, then $X_0(n)$ has irreducible reduction at p.

Some of the primes dividing $\mathrm{Disc}(\Delta_{n,n})$

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By monodromy considerations, there must be points of $X_0(n)$ or $X_1(n)$ in characteristic 0 which reduce to $\bar{c} = 0$ upon reduction modulo $\mathfrak{p}$ that collide $\rightsquigarrow p \mid \mathrm{Disc}(\Delta_{n,n})$.

Reduction and $p \mid (2^n - 1)$

Example

$n = 5$. Roots of $\Psi_n(x, 0)$ are ζ^i, $\zeta = e^{2\pi i/31}$, $1 \leq i \leq 30$.
6 orbits (in terms of i):

- 1, 2, 4, 8, 16
- 3, 6, 12, 24, 17
- 5, 10, 20, 9, 18
- 7, 14, 28, 25, 19
- 11, 22, 13, 26, 21
- 15, 30, 29, 27, 23

All collide modulo 31, remain distinct modulo all other primes

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$n = 6$. Roots of $\Psi_n(x, 0)$ are ζ^i, $\zeta = e^{2\pi i/63}$, $1 \leq i \leq 62$, $i \not\equiv 0 \pmod{21}$, $i \not\equiv 0 \pmod 9$. 9 orbits:

- 1, 2, 4, 8, 16, 32
- 3, 6, 12, 24, 48, 33
- 5, 10, 20, 40, 17, 34
- 7, 14, 28, 56, 49, 35
- 11, 22, 44, 25, 50, 37
- 13, 26, 52, 41, 19, 38
- 15, 30, 60, 57, 51, 39
- 23, 46, 29, 58, 53, 43
- 31, 62, 61, 59, 55, 47

Modulo 7: Seven orbits collide (the ones that aren't multiples of 3), and two other orbits collide (the ones that are) $\rightsquigarrow$ wild ramification!

$c = 0$ and $c = -2$

Similarly, roots of $\Phi_n(x, -2)$ are of the form $\zeta + \zeta^{-1}$, $\zeta \in \mu_{2^n-1} \cup \mu_{2^n+1}$.

Thus, except for certain small values of n: if $p \mid (2^n \pm 1)$, then $p \mid \mathrm{Disc}(\Delta_{n,n})$.

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Necessary criterion for good reduction: contribution to ramification divisors at $\bar{c} = 0$ and $\bar{c} = -2$ must be the same in characteristic 0 and characteristic p.

In many cases e.g. $(n, p) =$ $(6, 5), (6, 7), (6, 13), (7, 3), (7, 43), (7, 127), (8, 3), (8, 5), (8, 17), (8, 257)$, the only contribution to the ramification divisor comes from $\bar{c} = 0$ and $\bar{c} = -2$.

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Example

$\Delta_{5,5} \equiv c^5(c+2)^2 h(c) \pmod{31}$, where h is squarefree and not divisible by c or $c+2$ modulo 31. In the reduced curve, we have one six-cycle, so contribution is $6-1=5$ (tame ramification). This matches the exponent of c in $\Delta_{5,5}$, which is what we need.

Thank you

Thank you for your attention!