The Average Number of Divisors of the Euler Function

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Euler Function, Carmichael Function - Definitions and Notations

Let $n \geq 1$ be an integer. Denote by $\phi(n)$, $\lambda(n)$, the Euler Phi function and the Carmichael Lambda function, which output the order and the exponent of the group $(\mathbb{Z}/n\mathbb{Z})^*$ respectively.

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Let $n = p_1^{e_1} \cdots p_r^{e_r}$ be a prime factorization of n, then we can compute $\phi(n)$ and $\lambda(n)$ as follows:

$$\phi(n) = \prod_{i=1}^{r} \phi(p_i^{e_i}), \text{ and } \lambda(n) = \operatorname{lcm}\left(\lambda(p_1^{e_1}), \ldots, \lambda(p_r^{e_r})\right)$$

where $\phi(p_i^{e_i}) = p_i^{e_i - 1}(p_i - 1)$ and $\lambda(p_i^{e_i}) = \phi(p_i^{e_i})$ if $p_i > 2$ or $p_i = 2$ and $e_i = 1, 2$, and $\lambda(2^e) = 2^{e-2}$ if $e \geq 3$.

Definitions and Notations

We write $P_z = \prod_{p \leq z} p$. We also use the following restricted divisor functions:

$$\tau_z(n) := \prod_{\substack{p^e || n \\ p > z}} \tau(p^e), \quad \tau_{z,w}(n) := \prod_{\substack{p^e || n \\ z < p \leq w}} \tau(p^e), \quad \text{and} \quad \tau'_z(n) := \prod_{\substack{p^e || n \\ p \leq z}} \tau(p^e).$$

Moreover, for $n > 1$, denote by $p(n)$ the smallest prime factor of n.

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$$x \exp\left(\frac{1}{7}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log\log x}}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right) \le \sum_{n \le x} \tau(\lambda(n))$$

$$\le \sum_{n \le x} \tau(\phi(n)) \le x \exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log\log x}}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right).$$

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- Luca, Pomerance (LP, 2007): As $x \to \infty$,

$$\frac{1}{x}\sum_{n\le x}\tau(\lambda(n)) = o\left(\max_{y\le x}\frac{1}{y}\sum_{n\le y}\tau(\phi(n))\right).$$

F. Luca, C. Pomerance, *On the Average Number of Divisors of the Euler Function*, Publ. Math. Debrecen, 70/1-2 (2007), pp 125-148.

Main Results

- (Theorem 1.1) As $x \to \infty$, we have

$$\sum_{n \le x} \tau(\phi(n)) \ge \sum_{n \le x} \tau(\lambda(n)) \ge x \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log\log x}}(1+o(1))\right).$$

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- (Theorem 1.2) As $x \to \infty$,

$$\sum_{n \le x} \tau(\lambda(n)) = o\left(\sum_{n \le x} \tau(\phi(n))\right).$$

Heuristics

(Conjecture 1.1) As $x \to \infty$, we have

$$\sum_{n \le x} \tau(\lambda(n)) = x \exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log\log x}}(1+o(1))\right).$$

Thus, it is expected that both sums $\sum_{n \le x} \tau(\phi(n))$ and $\sum_{n \le x} \tau(\lambda(n))$ satisfy

$$x \exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log\log x}}(1+o(1))\right).$$

The Method - Theorem 1.1

- (Lemma 5 in LP) Let $A > 0$ and $1 < z \leq A\frac{\log x}{\log_2^4 x}$. Then

$$S_z(x) := \sum_{p \leq x} \frac{\tau_z(p-1)}{p} = c_1 \frac{\log x}{\log z} + O\left(\frac{\log x}{\log^2 z}\right).$$

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- (Corollary 2.1) Let $A > 1$ and $\log^{\frac{1}{A}} x \leq z \leq \log^A x$. Then as $x \to \infty$,

$$S_z(x) = c_1 \frac{\log x}{\log z} \left(1 + o(1)\right).$$

The Method - Theorem 1.1

• We use p_1, p_2, . . . , p_v to denote prime numbers. We define the following multiple sums for $2 \le v \le x$:

$$\mathfrak{T}_{v,z}(x) := \sum_{p_1 p_2 \cdots p_v \le x} \frac{\tau_z(p_1 - 1)\tau_z(p_2 - 1)\cdots\tau_z(p_v - 1)}{p_1 p_2 \cdots p_v},$$

and for $\mathbf{u} = (u_1, \ldots, u_v)$ with $1 \le u_i \le x$,

$$\mathfrak{T}_{\mathbf{u},v,z}(x) := \sum_{\substack{p_1 p_2 \cdots p_v \le x \\ \forall_i,\ p_i \equiv 1 \mod u_i}} \frac{\tau_z(p_1 - 1)\tau_z(p_2 - 1)\cdots\tau_z(p_v - 1)}{p_1 p_2 \cdots p_v},$$

Define $\mathbb{T}_v := \{(t_1, \ldots, t_v) : \forall_i,\ t_i \in [0,1],\ t_1 + \cdots + t_v \le 1\}$. We adopt the idea from Gauss' Circle Problem.

$$p_1 p_2 \cdots p_v \le x \Longleftrightarrow \frac{\log p_1}{\log x} + \frac{\log p_2}{\log x} + \cdots + \frac{\log p_v}{\log x} \le 1.$$

The Method - Theorem 1.1

• Let $v = \left\lfloor c\sqrt{\frac{\log x}{\log_2 x}} \right\rfloor$ for some positive constant c to be determined.
Use $\mathrm{vol}(\mathbb{T}_v) = \frac{1}{v!}$ to approximate

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This approximation allows a lower bound of

$$\sum_{n \le x} \frac{\tau(\lambda(n))}{n} \gg \exp\left(\sqrt{\frac{\log x}{\log_2 x}}\,(2c + c\log c_1 - 2c\log c + o(1))\right).$$

Maximizing above by the first derivative, the optimal choice for c is $e^{-\frac{\gamma}{2}}$.
This proves Theorem 1.1.

The Method - Theorem 1.2

- (Lemma 4.1) For any $2 \le y \le x$, we have

$$\sum_{n \le \frac{x}{y}} \frac{\tau(\phi(n))}{n} \ll \frac{\log^5 x}{x} \sum_{n \le x} \tau(\phi(n)).$$

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Define $\mathcal{E}_1(x)$, $\mathcal{E}_2(x)$ and $\mathcal{E}_3(x)$:

$$\mathcal{E}_1(x) := \{n \leq x : 2^k | n \text{ or there is a prime } p | n \text{ with } p \equiv 1 \bmod 2^k\},$$

$$\mathcal{E}_2(x) := \{n \leq x : \omega(n) \leq \omega\},$$

and

$$\mathcal{E}_3(x) := \{n \leq x\} - (\mathcal{E}_1(x) \cup \mathcal{E}_2(x)).$$

Use Lemma 4.1 to estimate sums $\sum \tau(\lambda(n))$ over the above three sets and obtain the estimate in Theorem 1.2.

The Binomial Model

For $z = \sqrt{\log x}$, let

$$\frac{\tau_{z,z^2}(\mathrm{lcm}(p_1 - 1, p_2 - 1, \ldots, p_v - 1))}{\tau_{z,z^2}(p_1 - 1)\tau_{z,z^2}(p_2 - 1)\cdots\tau_{z,z^2}(p_v - 1)}.$$

Let the number X_q of primes $p_1, \ldots, p_v$ such that $q|p_i - 1$. We model X_q by a binomial distribution with parameters v (number of trials), $\frac{2}{q}$ (probability of success).

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Thank you for listening!