Western Number Theory Problems, 17 & 19 Dec 2016

for distribution prior to 2017 (Pacific Grove) meeting

Edited by Gerry Myerson based on notes by Kjell Wooding

Summary of earlier meetings & problem sets with old (pre 1984) & new numbering.

1967 Berkeley	1968 Berkeley	1969 Asilomar	
1970 Tucson	1971 Asilomar	1972 Claremont	72:01–72:05
1973 Los Angeles	73:01–73:16	1974 Los Angeles	74:01–74:08
1975 Asilomar	75:01–75:23		
1976 San Diego	1–65 i.e., 76:01–76:65		
1977 Los Angeles	101–148 i.e., 77:01–77:48		
1978 Santa Barbara	151–187 i.e., 78:01–78:37		
1979 Asilomar	201–231 i.e., 79:01–79:31		
1980 Tucson	251–268 i.e., 80:01–80:18		
1981 Santa Barbara	301–328 i.e., 81:01–81:28		
1982 San Diego	351–375 i.e., 82:01–82:25		
1983 Asilomar	401–418 i.e., 83:01–83:18		
1984 Asilomar	84:01–84:27	1985 Asilomar	85:01–85:23
1986 Tucson	86:01–86:31	1987 Asilomar	87:01–87:15
1988 Las Vegas	88:01–88:22	1989 Asilomar	89:01–89:32
1990 Asilomar	90:01–90:19	1991 Asilomar	91:01–91:25
1992 Corvallis	92:01–92:19	1993 Asilomar	93:01–93:32
1994 San Diego	94:01–94:27	1995 Asilomar	95:01–95:19
1996 Las Vegas	96:01–96:18	1997 Asilomar	97:01–97:22
1998 San Francisco	98:01–98:14	1999 Asilomar	99:01–99:12
2000 San Diego	000:01–000:15	2001 Asilomar	001:01–001:23
2002 San Francisco	002:01–002:24	2003 Asilomar	003:01–003:08
2004 Las Vegas	004:01–004:17	2005 Asilomar	005:01–005:12
2006 Ensenada	006:01–006:15	2007 Asilomar	007:01–007:15
2008 Fort Collins	008:01–008:15	2009 Asilomar	009:01–009:20
2010 Orem	010:01–010:12	2011 Asilomar	011.01–011.16
2012 Asilomar	012:01–012:17	2013 Asilomar	013.01–013.13
2014 Pacific Grove	014:01–014:11	2015 Pacific Grove	015:01–015:15
2016 Pacific Grove	016:01–016:14		

COMMENTS ON ANY PROBLEM WELCOME AT ANY TIME

Department of Mathematics,
Macquarie University,
NSW 2109 Australia
gerry.myerson@mq.edu.au
Australia-2-9850-8952 fax 9850-8114

016:01 (Alessandro Rezende de Macedo) Let $\sigma_k(n) = \sum_{d|n} d^k$. How does σ_k fail to be injective, as a function of k?

Remarks: 1. Carl Pomerance noted that Erdős, On the normal number of prime factors of $p-1$ and some related problems concerning Euler's φ-function, Quart. J. Math., Oxford Ser. 6 (1935) 205–213, Zentralblatt 12, 149, available as 1935-08 at https://www.renyi.hu/~p_erdos/Erdos.html, used shifted primes to show that $\phi(n)$ is far from injective. Carl notes that the same proof, with -1 changed to $+1$, works for σ_1, and suggested this would be hard to do for $k > 1$—you would need to show something like p^2+1 is smooth for a positive proportion of primes p. See also C. Pomerance, Two methods in elementary analytic number theory, in R. A. Mollin, ed., Number Theory and Applications, Kluwer Academic Publishers, Dordrecht 1989, pp. 135–161.

2. We note that
$\sigma_2(6) = \sigma_2(7)$ (and $\sigma_2(6n) = \sigma_2(7n)$ for all n with $\gcd(n, 42) = 1$);
$\sigma_2(24) = \sigma_2(26)$ (and $\sigma_2(24n) = \sigma_2(26n)$ for $\gcd(n, 78) = 1$);
$\sigma_2(40) = \sigma_2(47)$ (and $\sigma_2(40n) = \sigma_2(47n)$ for $\gcd(n, 470) = 1$);
$\sigma_2(120) = \sigma_2(130) = \sigma_2(141)$; $\sigma_2(136) = \sigma_2(157)$; $\sigma_2(186) = \sigma_2(215) = \sigma_2(217)$;
$\sigma_2(230) = \sigma_2(249)$; and so on.

Pairs m, n with $\sigma_3(m) = \sigma_3(n)$ are tabulated at http://oeis.org/A131907/a131907.txt, due to Max Alekseyev. The smallest entry is $\sigma_3(184926) = \sigma_3(194315) = 7401260364550416$. It follows that the set of m for which there exists $n > m$ with $\sigma_3(m) = \sigma_3(n)$ has positive lower density.

016:02 (David Bailey) Let

$$w_n(t) = \sum_{m_1,\ldots,m_n \geq 1} \frac{1}{m_1^t m_2^t \cdots m_n^t (m_1 + \cdots + m_n)^t}$$

continued analytically to a meromorphic function on the plane.

1. Are there rationals $r_1, \ldots, r_k$ such that $w_n'(0) = \pm \log 2\pi + \sum_1^k r_j \zeta'(-2j)$?

2. If so, what are the rationals?

Remark: This has been settled for the first two cases, and there is numerical evidence for many more.

016:03 (Max Alekseyev, via Gerry Myerson) Let $n_1 = 3$, and for $k \geq 1$ let $n_{k+1} = 2^{n_k - 1} + 5$. Then we have $n_k \mid n_{k+1}$ for $k = 1, 2, 3, 4$. Does this divisibility hold for all k?

The question appears, with some discussion, at http://mathoverflow.net/questions/251717 and also at http://math.stackexchange.com/questions/1956027 and https://artofproblemsolving.com/community/c6h1316016p7069861 and (in Russian) at http://dxdy.ru/topic111861.html

Remark: Carl Pomerance suggests looking at n such that $n \mid 2^{n-1} + 5$ to see what patterns there may be. http://oeis.org/A245594 (contributed by Max Alekseyev) tabulates n such that $n \mid 2^n + 10$. From this, the n such that $n \mid 2^{n-1} + 5$ start with
$1, 3, 9, 161, 261, 5727, 12127, 577738261, 45019843643, 142046769201, 2247950127743$.

016:04 (Carl Pomerance) Let

$$S = \left\{ n : n \text{ divides } \binom{2n}{n} \right\}$$

This set is infinite, but misses a positive proportion of positive integers. In particular, it misses all those n with a prime factor exceeding $\sqrt{2n}$, a set of density $\log 2$. On the other hand, if $n = pq$ where p and q are prime and $1.5p < q < 2p$, then n is in S. Does S contain a positive proportion of the positive integers? Does S have a density?

Remarks: 1. These numbers are tabulated at http://oeis.org/A014847

2. Pante Stanica conjectures that for $n \geq 3700$ we have

$$\frac{n}{(\log\log n)^3} \leq \#S \leq \frac{n}{(\log\log n)^2}$$

016:05 (Andrew Shallue) Pomerance, On the distribution of pseudoprimes, Math. Comp. 37 (1981) 587–593 (see display (15), see also reference to Pomerance paper at **016:01**), proved that, for n fixed, and for x sufficiently large,

$$\#\{ m \leq x : \lambda(m) = n \} \leq xL(x)^{-1+o(1)}$$

Here $\lambda(m) = \mathrm{lcm}_{p|m}(p-1)$ is Carmichael's function, and

$$L(x) = \exp\left(\frac{\log x \log\log\log x}{\log\log x}\right)$$

Also,

$$\#\{ m \leq x : \lambda_2(m) = n \} \leq xL(x)^{-1+o(1)}$$

where $\lambda_2(m) = \mathrm{lcm}_{p|m}(p^2-1)$.

Can one improve this, perhaps on average?

016:06 (Neville Robbins) A *nosolo partition* of n is a partition in which each part occurs at least twice. The number of nosolo partitions of n equals the number of partitions of n into parts not $\pm 1 \bmod 6$. For $d \geq 3$, find a similar equality for the number of partitions of n in which each part occurs at least d times.

Remarks: 1. The number of nosolo partitions is tabulated at http://oeis.org/A007690

2. The number of nosolo partitions is also the number of partitions into parts, each larger than one, and differing by at least two.

3. Simon Rubinstein-Salzedo suggests consulting Bruce Berndt's book, Number Theory in the Spirit of Ramanujan.

016:07 (Sungjin Kim) Are the matrices

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{pmatrix}, \ldots$$

all nonsingular? The entries here are the first $4, 9, 16, \ldots$ primes, entered in order, left to right and then top to bottom.

Remarks: 1. This is essentially http://math.stackexchange.com/questions/2047879/matrix-generated-by-prime-numbers proposed by user Widawensen, but there the primes are taken to start from 1.

2. Nonsingularity has been verified for both versions of the question up to size 200×200 by user Peter.

3. Renate Scheidler asked whether there is anything special here about primes, and whether one could just choose randomly increasing integers, noting that a random integer matrix will be nonsingular. [I suppose it depends on what's meant by "random", e.g.,

$$\begin{pmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 101 \end{pmatrix}$$

is singular.]

4. Simon Rubinstein-Salzedo asks whether the determinants are always negative, and whether they are increasing in absolute value.

016:08 (Simon Rubinstein-Salzedo) A number n is *practical* if each number less than n is a sum of distinct divisors of n. For $k \geq 4$, do there exist infinitely many n such that $n, n+2, \dots, n+2(k-1)$ are all practical?

Remark: This is true for $k = 3$ (G. Melfi, On two conjectures about practical numbers, J. Number Theory 56 (1996) 205–210, MR 1370203 (96i:11106)).

016:09 (Bart Goddard) Is there any arrangement of the first n^2 primes in an $n \times n$ matrix such that the matrix is singular? (Compare 016:07)

Solution: Kevin McGown found $\det A = 0$ for

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 13 & 11 & 7 \\ 17 & 19 & 23 \end{pmatrix}$$

016:10 (Christian Ballot) Let $\alpha = (1 + \sqrt{5})/2$. Consider a tree built as follows. The first level consists of the number 0. Each number r at each level has the two descendants $r + 1$ and $r\alpha$, except that we prune any number that has already appeared. Thus, the second level has just the number 1; the third level has 2 and α; the fourth level has 3, 2α, and $\alpha + 1$ (since $\alpha^2 = \alpha + 1$). The number of numbers on each level of the tree, G_n, forms a sequence starting $1, 1, 2, 3, 5, 8$, but $G_7 = 12$, so it's not the Fibonaccis. It is conjectured that $G_{n+3} = G_{n+2} + G_n$ for $n \geq 13$.

Remark: The sequence is tabulated at http://oeis.org/A252864

016:11 (Carter Smith) Define $S : \mathbf{N} \to \mathbf{R}^2$ by $S(n) = (\sqrt{n}, 2\pi\phi n)$ in polar coordinates, $S(n) = (\sqrt{n}\cos 2\pi\phi n, \sqrt{n}\sin 2\pi\phi n)$ in rectangular, where ϕ is the golden ratio. Then the points corresponding to prime values of n lie on specific rays. As ϕ varies, the number of rays varies, and the picture changes, drastically.

Remarks: 1. Andrew Shallue references Matt Boelkins' plenary talk, Fibonacci's Garden, at the meeting of the Illinois Section of the MAA in 2016.

2. Sungjin Kim suggests the pictures for rational and irrational ϕ will be very different.

016:12 (Paul Young) For x in $\mathbf{C}$, define

$$D_n(x) = \det \begin{pmatrix} \binom{x}{2} & \binom{x}{1} & 0 & \dots & \dots \\ \binom{x}{3} & \binom{x}{2} & \binom{x}{1} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{x}{n+1} & \dots & \dots & \dots & \binom{x}{2} \end{pmatrix}$$

It is known that if $n > 1$ is odd, and $d \mid (n-2)$, then $D_n(d) = 0$. Show that for complex x if $0 < |x| < 1$ then $D_n(x) \neq 0$.

Remark: This is known to be true if $x < (1 + \log n)^{-1}$, and has been verified computationally for all $n < 100$.

016:13 (Elie Alhajjar) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries, $bc \neq 0$, $\Delta = |ad - bc| \geq 2$. Let $A^n = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Is there a formula for $\gcd(\alpha, \gamma)$ in terms of a, b, c, d, n? Note that $\gcd(\alpha, \gamma)$ divides Δ^n.

016:14 (Bernardo Recamán Santos, via Gerry Myerson) Is it possible to completely tile a square with different rectangles of integer sides but all with the same area?

Remarks: 1. This is http://mathoverflow.net/questions/220567/tiling-a-square-with-rectangles

2. Without the integer restriction, there are many solutions. One has seven rectangles, each of area 1/7, with sides in $\mathbf{Q}(\sqrt{19})$.

3. Ed Pegg, Jr. suggests a number of candidate squares, e.g., a square of side 3960 has area 33×475200, and there are 40 integer rectangles of area 475200.