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The aliquot constant

Carl Pomerance

Dartmouth College (emeritus)

University of Georgia (emeritus)

The sum-of-proper-divisors function

Let $s(n)$ be the sum of the *proper* divisors of n:

For example:

$$s(10) = 1 + 2 + 5 = 8,$$
$$s(11) = 1,$$
$$s(12) = 1 + 2 + 3 + 4 + 6 = 16.$$

In modern notation: $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum of all of n's natural divisors.

Pythagoras noticed that $s(6) = 1 + 2 + 3 = 6$
If $s(n) = n$, we say n is *perfect*.

And amazingly, he noticed that

$$s(220) = 284, \quad s(284) = 220.$$

By iterating s, Pythagoras was looking at the first dynamical system!

That is, keep iterating s until one gets to a cycle, to 0, or ...?

A sequence under s-iteration is known as an aliquot sequence:

$10 \to 8 \to 7 \to 1$

$12 \to 16 \to 15 \to 9 \to 4 \to 3 \to 1$

$14 \to 10 \ldots$

$18 \to 21 \to 11 \to 1$

$20 \to 22 \to 14 \ldots$

$24 \to 36 \to 55 \to 17 \to 1$

$25 \to 6 \to 6$

$26 \to 16 \ldots$

$28 \to 28$

$30 \to 42 \to 54 \to 66 \to 78 \to 90 \to 144 \to 259 \to 45 \to 33 \to 15 \ldots$

$\vdots$

The **Catalan–Dickson** conjecture: Every aliquot sequence is bounded.

The **Guy–Selfridge** counter conjecture: Most aliquot sequences starting from an even number are unbounded.

No unbounded aliquot sequence is known, the least starter in doubt is 276, having been pursued for over two thousand iterations. Computations have bogged down where the numbers involved have about 210 digits.

If p, q are different primes and $n = p + q + 1$, then $n = s(pq)$ is a value of s. A slightly stronger form of **Goldbach**'s conjecture implies that every even number starting with 8 is the sum of two different odd primes p, q, so this conjecture implies that starting from any odd number $n \geq 9$ there is an infinite sequence $\cdots > n_2 > n_1 > n_0 = n$, where $s(n_i) = n_{i-1}$.

In 1990, **Erdős, Granville, P, Spiro** showed that this argument works for "almost all" odd numbers n. In particular there are arbitrarily long decreasing "aliquot" sequences.

Lenstra (1975):
There are arbitrarily long increasing aliquot sequences

$$n < s(n) < s(s(n)) < \cdots < s_k(n).$$

Erdős (1976): *In fact, for each fixed k, if $n < s(n)$, then almost surely the sequence continues to increase for $k-1$ more steps.*

(A corollary: The amicable numbers have asymptotic density 0, since if n is the smaller member of a pair, we have $s(s(n)) = n < s(n)$.)

Recently **Bosma** did a statistical study of aliquot sequences with starting numbers below 10^6. About one-third of the even starters are still open and running beyond 10^{99}. Evidence for **Guy–Selfridge**? But: he and **Kane** (2012) found the geometric mean of the ratios $s(2n)/2n$ asymptotically, finding it slightly below 1. Evidence for **Catalan–Dickson**?

They showed that

$$\frac{2}{x} \sum_{\substack{n \le x \\ n \text{ even}}} \log\left(\frac{s(n)}{n}\right) \sim \lambda < -0.03.$$

Bosma & Kane called the number

$$\lambda = \lim_{x\to\infty} \frac{2}{x} \sum_{\substack{n\le x\\ n \text{ even}}} \log\left(\frac{s(n)}{n}\right)$$

the *aliquot constant*. They computed the expression at $x = 3.923 \times 10^9$ finding it to be $-0.03325\,97045\ldots$.

Mosunov computed the sum at $x = 2^{40}$ and found the value $-0.03325\,94805\ldots$.

A few months ago I proved the limit converges with a power savings, i.e., the error from the limit at x is of the shape x^{-c} for some $c > 0$. I also rigorously computed λ to 13 decimal places:

$$\lambda = -0.03325\,94844\,693\ldots.$$

However, the Catalan–Dickson conjecture involves iterating the function $s(n)$, while the aliquot constant deals only with the first iterate.

P (2016):

- *The asymptotic geometric mean of the ratios $s(s(2n))/s(2n)$ is also* e^{λ}.

- *Assuming a conjecture of* **Erdős, Granville, P, & Spiro**, *for each fixed k, there is a set A_k of asymptotic density* 1 *such that the asymptotic geometric mean of $s_k(2n)/s_{k-1}(2n)$ on A_k is also* e^{λ}.

The conjecture mentioned:
If E has asymptotic density 0*, so does $s^{-1}(E)$.*

Pollack, P, Thompson (2017): This conjecture holds in the case that E is very sparse, with counting function $O(x^{1/2+\epsilon})$.

There are some recent numerical experiments by **Chum, Guy, Jacobson, & Mosunov** with analogs of the aliquot constant for higher iterates. Computing to 2^{37} they found the j-iterate analog as

$$\begin{aligned}
j = 1: &\quad -0.03326\\
j = 2: &\quad -0.03706\\
j = 3: &\quad -0.01849\\
j = 4: &\quad -0.01205\\
j = 5: &\quad -0.00411\\
j = 6: &\quad +0.00145\\
j = 7: &\quad +0.00779\\
j = 8: &\quad +0.01297\\
j = 9: &\quad +0.01854\\
j = 10: &\quad +0.02339.
\end{aligned}$$

How is the aliquot constant $\lambda = \lim_{x\to\infty} \frac{2}{x} \sum_{\substack{n\le x \\ n \text{ even}}} \log\left(\frac{s(n)}{n}\right)$ calculated?

Note that for $n > 1$,

$$\log\left(\frac{s(n)}{n}\right) = \log\left(\frac{\sigma(n) - n}{n}\right) = \log\left(\frac{\sigma(n)}{n}\right) + \log\left(1 - \frac{n}{\sigma(n)}\right)$$
$$= \log\left(\frac{\sigma(n)}{n}\right) - \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{n}{\sigma(n)}\right)^j.$$

The function $\log(\sigma(n)/n)$ is additive and it is easy to find its average order:

$$\sum_{n\le x} \log\left(\frac{\sigma(n)}{n}\right) = \alpha x + O(\log\log x),$$

where $\alpha = 0.44570\,89138\,58165\,8 \pm 7 \times 10^{-16}$.

Though most of the work is in figuring the average order of the correction terms $\frac{1}{j}(n/\sigma(n))^j$, for this talk I'll mainly concentrate on α.

Note that

$$\log\left(\frac{\sigma(p^a)}{p^a}\right) = \sum_{i=1}^{a}\left(\log\left(\frac{\sigma(p^i)}{p^i}\right) - \log\left(\frac{\sigma(p^{i-1})}{p^{i-1}}\right)\right) = \sum_{i=1}^{a}\Lambda_\sigma(p^i),$$

where $\Lambda_\sigma(p^i) = \log(1 + 1/(\sigma(p^i) - 1))$. For a non-prime-power, let $\Lambda_\sigma(d) = 0$. Thus,

$$\log\left(\frac{\sigma(n)}{n}\right) = \sum_{d\,|\,n}\Lambda_\sigma(d).$$

And so,

$$\sum_{n\le x} \log\left(\frac{\sigma(n)}{n}\right) = \sum_{n\le x}\sum_{d\,|\,n} \Lambda_\sigma(d) = \sum_{d\le x} \Lambda_\sigma(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

Removing the floor symbol and summing to infinity creates only small errors, and we are left with computing

$$\alpha = \sum_{d\ge 1} \frac{\Lambda_\sigma(d)}{d}.$$

This sum lives only on the primes and prime powers, and the most important (largest) terms are when d is prime:

$$\frac{\Lambda_\sigma(p)}{p} = \frac{\log(1+1/p)}{p} \sim \frac{1}{p^2}.$$

More generally, $\Lambda_\sigma(p^i)/p^i$ is of magnitude $1/p^{2i}$ so that if we sum for $p \le x$, the error in truncating at this point will be of magnitude $1/(x \log x)$.

We can accelerate the convergence by using that the prime terms with $i = 1$ are $\sim 1/p^2$, so the sum of these terms should converge similarly as

$$\log(\zeta(2)) = \sum_p -\log(1 - 1/p^2) = \log(\pi^2/6).$$

The upshot is that

$$\alpha = \log(\pi^2/6) + \sum_p \left(\frac{\log(1 + 1/p)}{p} + \log(1 - 1/p^2) \right) + \sum_{p^i,\, i>1} \frac{\Lambda_\sigma(p^i)}{p^i}.$$

And truncating *these* sums at x creates an error of magnitude $1/(x^2 \log x)$.

Using Mathematica and letting x be the one-millionth prime, I was able to compute α to 15 decimal places.

Similar tricks were used for the remainder of the calculations, so in the end, very little computing power was used.

Thank You!