A few words on some things

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A little poetry to get us started

Alexandria, Greece ~ 250AD

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‘Here lies Diophantus,’ the wonder behold.
Through art algebraic, the stone tells how old:
‘God gave him his boyhood one-sixth of his life,
One twelfth more as youth while whiskers grew rife;
And then yet one-seventh ere marriage begun;
In five years there came a bouncing new son.
Alas, the dear child of master and sage
After attaining half the measure of his father’s life,
chill fate took him.
After consoling his fate by the science of numbers for four years,
he ended his life.’

interuallum numerorum 2. minor autem 1 N. atque ideo maior 1 N. + 2. Oportet itaque 4 N. + 4. triplos esse ad 2. & adhuc superaddere 10. Ter igitur 2. adscitis vnitatibus 10. æquatur 4 N. + 4. & fit 1 N. 3. Erit ergo minor 3. maior 5. & satisfaciunt quæstioni.

ςˢ ἑνός. ὁ ἄρα μείζων ἔσται ςˢ ἑνὸς μˣ β̄. δεήσει ἄρα ἀριθμοὺς δ̄ μονάδας δ̄ τριπλασίονας εἶναι μˣ β̄. καὶ ἔτι ὑπερέχειν μˣ ῑ. τρὶς ἄρα μονάδες β̄ μετὰ μˣ ῑ. ἴσαι εἰσὶν ςˢ δ̄ μονάσι δ̄. καὶ γίνεται ὁ ἀριθμὸς μˣ γ̄. ἔσται ὁ μὲν ἐλάσσων μˣ γ̄. ὁ δὲ μείζων μˣ ε̄. καὶ ποιοῦσι τὸ πρόβλημα.

IN QVAESTIONEM VII.

CONDITIONIS appositæ eadem ratio est quæ & appositæ præcedenti quæstioni, nil enim aliud requirit quàm vt quadratus interualli numerorum sit minor interuallo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quod continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. $\frac{16}{5}$ Erit igitur alter quadratorum $\frac{256}{25}$. alter verò $\frac{144}{25}$ & vtriusque summa est $\frac{400}{25}$ seu 16. & vterque quadratus est.

ΤΟΝ ἐπιταχθέντα τετράγωνον διελεῖν εἰς δύο τετραγώνους. ἐπιτετάχθω δὴ τὸν ιϛ̄ διελεῖν εἰς δύο τετραγώνους. καὶ τετάχθω ὁ πρῶτος δυνάμεως μιᾶς. δεήσει ἄρα μονάδας ιϛ̄ λεῖψει δυνάμεως μιᾶς ἴσας εἶναι τετραγώνῳ. πλάσσω τὸν τετράγωνον ἀπὸ ςˢ. ὅσων δή ποτε λεῖψει τοσούτων μˣ ὅσων ἐστὶν ἡ τῶν ιϛ̄ μˣ πλευρά. ἔστω ςˢ β̄ λεῖψει μˣ δ̄. αὐτὸς ἄρα ὁ τετράγωνος ἔσται δυνάμεων δ̄ μˣ ιϛ̄ λεῖψει ςˢ ιϛ̄. ταῦτα ἴσα μονάσι ιϛ̄ λεῖψει δυνάμεως μιᾶς. κοινὴ προσκείσθω ἡ λεῖψις, καὶ ἀπὸ ὁμοίων ὅμοια. δυνάμεις ἄρα ε̄ ἴσαι ἀριθμοῖς ιϛ̄. καὶ γίνεται ὁ ἀριθμὸς ιϛ̄ πέμπτων. ἔσται ὁ μὲν σνϛ̄ εἰκοστοπέμπτων. ὁ δὲ ρμδ̄ εἰκοστοπέμπτων, καὶ οἱ δύο συντεθέντες ποιοῦσι ῡ εἰκοστόπεμπτα, ἤτοι μονάδας ιϛ̄. καὶ ἔστιν ἑκάτερος τετράγωνος.

OBSERVATIO DOMINI PETRI DE FERMAT.

CVbum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16 diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. $\frac{16}{5}$ erit

ΕΣΤΩ δὴ πάλιν τὸν ιϛ̄ τετράγωνον διελεῖν εἰς δύο τετραγώνους. τετάχθω πάλιν ἡ τοῦ πρώτου πλευρὰ ςˢ ἑνός, ἡ δὲ τοῦ ἑτέρου ςˢ ὁποσωνδήποτε λεῖψει μˣ ὅσων ἐστὶν ἡ τοῦ διαιρουμένου πλευρά. ἔστω δὴ ςˢ β̄ λεῖψει μˣ δ̄. ἔσονται οἱ τετράγωνοι ὃς μὲν δυνάμεως μιᾶς, ὃς δὲ δυνάμεων δ̄ μˣ ιϛ̄ λεῖψει ςˢ ιϛ̄. βούλομαι τοὺς δύο λοιπὸν συντεθέντας ἴσους εἶναι μˣ ιϛ̄. δυνάμεις ἄρα ε̄ μˣ ιϛ̄ λεῖψει ςˢ ιϛ̄ ἴσαι μˣ ιϛ̄. καὶ γίνεται ὁ ἀριθμὸς ιϛ̄ πέμπτων.

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$$F_t(X, Y) = c.$$

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$m = 2:$

$$3^X + 4^Y = 5^Z$$

Theorem (He & Togbé, 2009)

Let $m \in \mathbb{Z}^+$, $m \geq 4$. Then the equation

$$(2m-1)^X + (2m)^Y = (2m+1)^Z$$

has no positive integer solutions X, Y, Z.

Some snippets of the proof

Suppose $m \geq 4$, and x, y, $z \in \mathbb{Z}^+$ such that

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$$(2m-1)^x + 2m = (2m+1)^z$$

Linear forms in logarithms

$$b_1 \log \alpha_1 + b_2 \log \alpha_2 + \ldots + b_n \log \alpha_n$$

with $b_i \in \mathbb{Z}$ and $\alpha_i \in \overline{\mathbb{Q}}$.

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$$D = [\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) : \mathbb{Q}]/[\mathbb{R}(\alpha_1, \alpha_2, \ldots, \alpha_n) : \mathbb{R}]$$

For any $\alpha \in \overline{\mathbb{Q}}$,

$$d = [\mathbb{Q}(\alpha) : \mathbb{Q}],$$

$$h(\alpha) = \frac{1}{d}\left(\log|a_d| + \sum_{j=1}^{d} \max\{\log|\sigma_j(\alpha)|, 0\}\right),$$

$$\log A_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\}.$$

In this case, b_1, b_2, α_1, $\alpha_2 \in \mathbb{Z}^+$, with $\alpha_i > 1$, then

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Theorem (Laurent, Mignotte, Nesterenko (1995))

Let

$$\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2$$

and

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

If $\alpha_1, \alpha_2, \log \alpha_1, \log \alpha_2 \in \mathbb{R}^+$ *and* $\Lambda \neq 0$, *then*

$$\log |\Lambda| \geq -24.34 \left(\max\left\{\log b' + 0.14, 21\right\}\right)^2 \log A_1 \log A_2.$$

Recall

$$(2m-1)^x + 2m = (2m+1)^z \Rightarrow 1 + \frac{2m}{(2m-1)^x} = \frac{(2m+1)^z}{(2m-1)^x}.$$

Let $\Lambda = z\log(2m+1) - x\log(2m-1)$.

Using Laurent, Mignotte, and Nesterenko (1995):

$$\begin{aligned}\log|\Lambda| \quad &\geq -24.34\left(\log\left(\frac{2x}{\log(2m+1)} + 1.54\cdot 10^{-167}\right) + 0.14\right)^2 \\ &\times \log(2m+1)\log(2m-1)\end{aligned}$$

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Hence, no large solutions to $(2m-1)^x + (2m)^y = (2m+1)^z$.

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