Additive Factorials (mod p) (where p is a positive, even integer)

Amy Feaver*, Zack Baker

West Coast Number Theory 2017

Give me feedback!

- Read "Imaginary Multiquadratic Fields of Class Number Dividing 2^m" (with Anna Puskas)

- If you have thoughts about today's presentation, be brutally honest!

Setup - Example

$\mathbb{Z}/6\mathbb{Z}$, additive group

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$1\ddagger = 1 + 4 + 3 + 2 = 4$
$3\ddagger = 3 + 2 = 5$, etc.

Setup-Example

If $(2, 3, 4, 1, 5, 0)$ the plower set is

$$\{2\dagger, 3\dagger, 4\dagger, 1\dagger, 5\dagger, 0\dagger\} = \{2, 5, 3, 4, 3, 3\} = \{2, 3, 4, 5\}.$$

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If $(0, 1, 4, 3, 2, 5)$, the plower set is $\{0, 1, 5, 2, 4, 3\} = \mathbb{Z}/n\mathbb{Z}$

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$(0,1,4,3,2,5)$ is called a *plowmutation* of $\mathbb{Z}/6\mathbb{Z}$

Setup - Example

The plowmutations of $\mathbb{Z}/6\mathbb{Z}$ are:

(0,1,4,3,2,5)

(0,2,5,3,1,4)

(0,4,1,3,5,2)

(0,5,2,3,4,1)

Notation Vote?

I might like the "bling" better than the "plow":

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$5\ast\!\!\!\!\!\;! = 5 \ast 4 \ast 3 \ast 2 \ast 1$ where $\ast$ is the group operation.

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We want to count the number of plowmutations of $(0, 1, 2, ..., n-1)$

Number of Orderings

n	Number of plowmutations
2	1
3	0
4	2
5	0
6	4
7	0
8	24
9	0
10	288
11	0
12	3856

Number of Orderings

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For $2n$ we have sequence 1,2,4,24,288,3856,... where a_n is the number of plowmutations of $2n$.

Number of Orderings

A141599: Number of inequivalent difference sets for permutations of $2n$ with distinct differences is 1, 2, 4, 24, 288, 3856, 89328, 2755968,...

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Is (provably) the same sequence!

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Goal: Calculate more terms of this sequence.

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Method: Develop theory (me), multithread algorithm and use heavy machinary (Zack)

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(Not a conjecture.) The natural order is a plowmutation if and only if n is a power of 2

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$(0 \cdot a, g_1 \cdot a, g_2 \cdot a, ..., g_{n-1} \cdot a)$ is plowmutation iff $\gcd(a, n) = 1$

Data

	1	2	3	4	5	6	7	8	9
Position 1	43	29	43	29	0	29	43	29	43
Position 2	26	34	26	34	48	34	26	34	26
Position 3	30	34	30	34	32	34	30	34	30
Position 4	30	34	30	34	32	34	30	34	30
Position 5	30	26	30	26	64	26	30	26	30
Position 6	30	34	30	34	32	34	30	34	30
Position 7	30	34	30	34	32	34	30	34	30
Position 8	26	34	26	34	48	34	26	34	26
Position 9	43	29	43	29	0	29	43	29	43

Sequenceable Groups

A group G is called *sequencable* if there exists a sequence of elements of that group $g_1, g_2, g_3, \ldots$ such that

$$\{g_1,\ \ g_2 * g_1,\ \ g_3 * g_2 * g_1,\ \ g_4 * g_3 * g_2 * g_1, \ldots.\}$$

is equal to the whole group.

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Re-frame the problem: A group G is called *sequencable* if there exists an ordering on the elements $g_1 \prec g_2 \prec g_3 \prec \cdots$ such that

$$\{g_i \underset{\cdot}{*}\} = G.$$

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"What (quilting) cirlces can be squared?" (Beth Malmskog, Kathryn Haymaker, Gretchen Matthews). To appear in math magazine.

Sequenceable Groups

(For problem session?) Is it possible to find an permutation on a group such that the set of all “blings” gives us a proper subgroup of the group?

There are no such permutations for $\mathbb{Z}/n\mathbb{Z}$

Current Status of this Project

Using a computer with 8 cores, Zack can re-preoduce the sequence in less than 24 hours including all of the tables of how often each element appears in each position.

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With more computing power we could easily push these calculations further, I think through the 11th term - we currently have 8).

The End

Questions, Comments, Criticisms, Concerns, Proofs?