West Coast Number Theory
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S-Euclidean Imaginary Quadratic Fields

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Imaginary quadratic number field

An imaginary quadratic number field is an extension of the rational numbers of degree 2 with negative discriminant. In general it is given by $K = \{a + b\sqrt{-d} \mid a, b \in \mathbb{Q}\}$

Ring of integers

In the imaginary quadratic case, the ring of integers is given by $\mathcal{O}_K = \{x + y\sqrt{-d} \mid x, y \in \mathbb{Z}\}$ when $-d \equiv 2, 3 \pmod 4$ or $\mathcal{O}_K = \{x + y\frac{1+\sqrt{-d}}{2}\}$ when $-d \equiv 1 \pmod 4$

The norm map

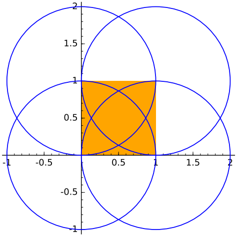
In the imaginary quadratic setting, the norm is given by $N(x + y\sqrt{-d}) = x^2 + dy^2$.

Euclidean number field

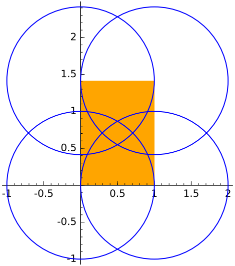
A number field is said to be Euclidean if for all $\xi \in K$, there exists $\gamma \in \mathcal{O}_K$ such that the $N(\xi - \gamma) < 1$

If K is Euclidean, then it has class number one. However, the converse is not true.

Examples of Euclidean Fields



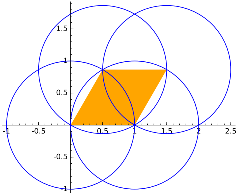
$\mathbb{Q}(\sqrt{-1})$



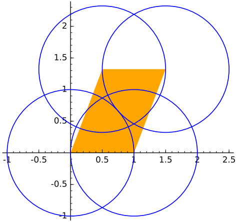
$\mathbb{Q}(\sqrt{-2})$

More Examples

$\mathbb{Q}(\sqrt{-3})$



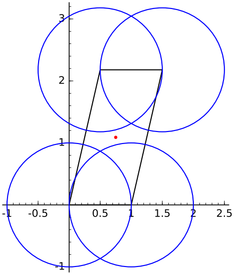
$\mathbb{Q}(\sqrt{-7})$



Failing to be Euclidean

When K fails to be Euclidean there exists some $\xi \in K$ such that for all $\alpha \in \mathcal{O}_K$ $N(\xi - \alpha) \geq 1$. For example if $d = -19$ and $\xi = \frac{3+\sqrt{-19}}{4}$ and we let $\alpha = a + b(\frac{1+\sqrt{-19}}{2})$ be arbitrary. Then

$$N(\xi - \alpha) = \frac{(3 + 4a + 2b)^2 + 19(1 - 2b)^2}{4} > \frac{19}{4} > 1$$



S-Integers

Let S be a finite set of rational primes containing 2. Let S_K denote the set of primes in K lying above primes in S, together with the infinite primes. The ring of S-integers is defined as

$$\mathcal{O}_{K,S} = \{\xi \in K \mid v(\xi) \geq 0 \ \ \forall v \notin S_K\}\,.$$

In our setting, we have

$$\mathcal{O}_{K,S} = \left\{ \frac{x + y\sqrt{-d}}{\tau} \mid x, y \in \mathbb{Z} \right\},$$

where τ is a product of primes in S.

S-Norm

We define the S norm as $N_S(x) = N(x) \prod_{v \in S_K \setminus S_\infty} |x|_v$.

This has the effect of deleting the primes in S from $N(x)$.

Example

Take K $= \mathbb{Q}(\sqrt{-10})$ with $S = \{2, 3\}$. Then the $N(\frac{4+6\sqrt{-10}}{7}) = \frac{376}{49} = \frac{2^3 * 47}{7*7}$. Now the $N_S(\frac{4+6\sqrt{-10}}{7}) = \frac{47}{49} = \frac{47}{7*7}$.

S-Euclidean

We say that K is S-Euclidean if for all $\xi \in K$ there exists $\gamma \in \mathcal{O}_{K,S}$ such that $N_S(\xi - \gamma) < 1$.

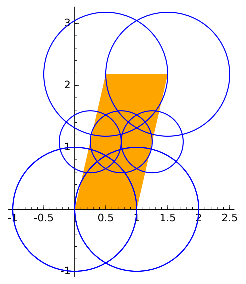
Lemma

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field. If $\xi \in K$ and $\gamma \in \mathcal{O}_{K,S}$ such that $\gamma = \frac{a+b\sqrt{-d}}{\tau}$ where $a, b \in \mathbb{Z}$ and τ is a product of primes in S then

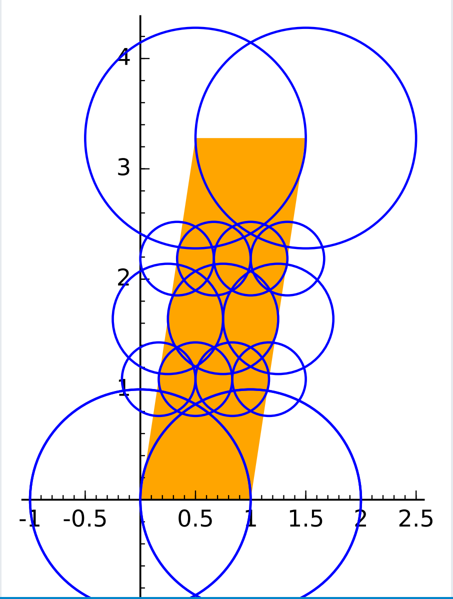
$$N_S(\xi - \gamma) \leq \tau^2 N(\xi - \gamma).$$

This allows us to draw circles of radius $\frac{1}{\tau}$, with centers at $\frac{a+b\sqrt{-d}}{\tau}$.

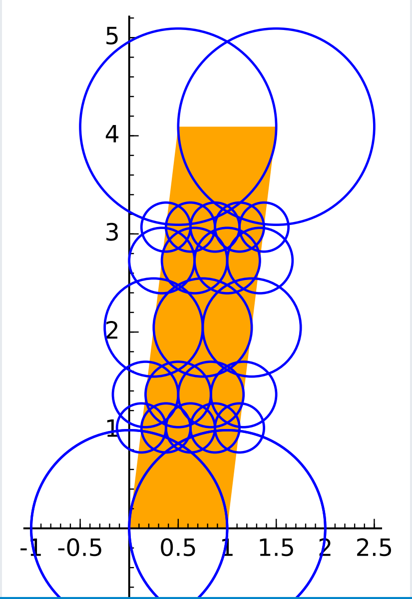
$K = \mathbb{Q}(\sqrt{-19})$ is S-Euclidean for $S = \{2\}$



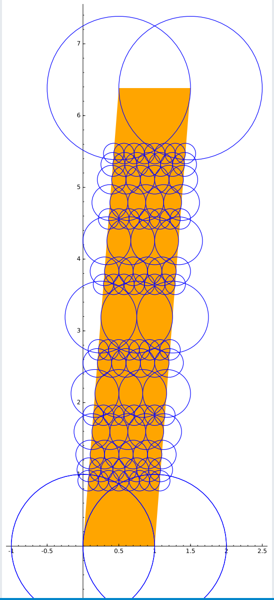
$K = \mathbb{Q}(\sqrt{-43})$ is S-Euclidean for $S = \{2, 3\}$



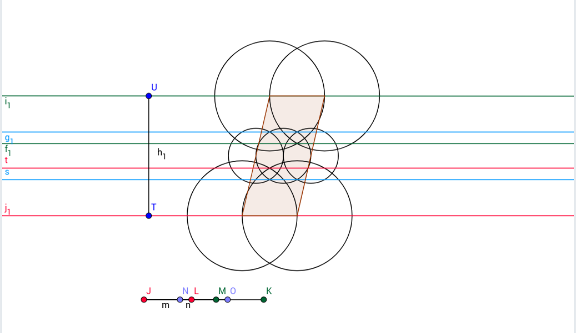
$K = \mathbb{Q}(\sqrt{-67})$ is S-Euclidean for $S = \{2, 3\}$



$K = \mathbb{Q}(\sqrt{-163})$ is S-Euclidean for $S = \{2, 3, 5, 7\}$



Algorithm



1. Instead of the whole circle create intervals from the points where neighboring circles of the same radius intersect.
2. If the intervals cover then the field is S-Euclidean.

Some S-Euclidean Fields found with the Algorithm

$S = \emptyset$:

$-1, -2, -3, -7, -11$

$S = \{2\}$:

$-5, -6, -15, -19, -23$

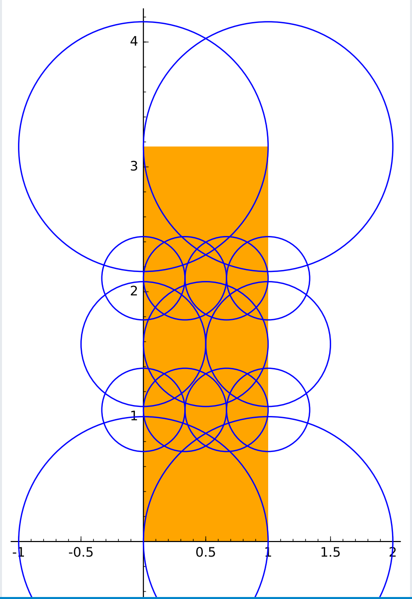
$S = \{2, 3\}$:

$-10, -13, -14, -17, -31, -35, -39, -43, -47, -51, -55, -59,$
$-67, -71$

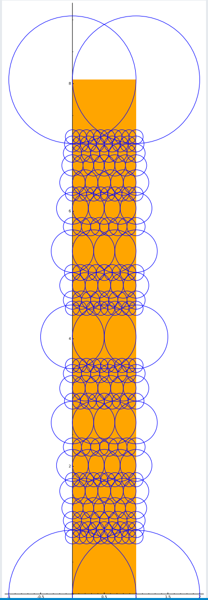
$S = \{2, 3, 5\}$:

$-21, -22, -26, -29, -30, -33, -34, -79, -83, -87, -91, -95,$
$-103, -107, -111, -115, -119, -123, -127, -131, -139, -143$

$\mathbb{Q}(\sqrt{-10})$ $S = \{2, 3\}$



$\mathbb{Q}(\sqrt{-65})$ $S = \{2, 3, 5, 7\}$



Theorem (H. – Moses)

Let K be an imaginary quadratic field with discriminant D and S be a set of rational primes. If S contains all all the primes up to $\sqrt{D}$, then K is S-Euclidean.

Proof

We need to show that for all $x \in [0, \frac{\sqrt{D}}{2}]$ there exist an $a, z \in \mathbb{Z}$ where $(a, z) = 1$ and $1 \leq a \leq z$ such that $|x - \frac{a\sqrt{D}}{2z}| < \frac{\sqrt{3}}{2z}$. Manipulating this inequality we get $|zy - a| < \frac{\sqrt{3}}{\sqrt{D}}$. Therefore, It suffices to show that for all $y \in [0, 1)$, there exists a $z \in \mathbb{Z}^+$ such that $\{yz\} < \frac{\sqrt{3}}{\sqrt{D}}$. Now we can run through $\{0y\}, \{y\}, \ldots \{(\lceil \frac{\sqrt{D}}{\sqrt{3}} \rceil + 1)y\}$. and see where these land in our interval. We can rewrite the interval $[0, 1) = [0, \frac{\sqrt{3}}{\sqrt{D}}) \cup [\frac{\sqrt{3}}{\sqrt{D}}, \frac{2\sqrt{3}}{\sqrt{D}}) \ldots [1 - \frac{\sqrt{3}}{\sqrt{D}}, 1)$.

Theorem (H. – Moses)

Let K be an imaginary quadratic field with discriminant D and S be a set of rational primes. If S contains all all the primes up to $\sqrt{D}$, then K is S-Euclidean.

Proof (continued)

We now have $\lceil \frac{\sqrt{D}}{\sqrt{3}} \rceil + 1$ items and only $\lceil \frac{\sqrt{D}}{\sqrt{3}} \rceil$ intervals. This implies that there must exist an $s, t \in \mathbb{Z}$ satisfying $1 \leq s < t \leq \lceil \frac{\sqrt{D}}{\sqrt{3}} \rceil$ such that $\{sy\}, \{ty\}$ differ by less than $\frac{\sqrt{3}}{\sqrt{D}}$. This means that if we let $z \geq \lceil \frac{\sqrt{D}}{\sqrt{3}} \rceil + 1$ we get what we wanted to show. Now this implies that inverting all primes up to or beyond $\lceil \frac{\sqrt{D}}{\sqrt{3}} \rceil + 1$ the field will become S-Euclidean. Since $\lceil \frac{\sqrt{D}}{\sqrt{3}} \rceil + 1 \leq \sqrt{D}$ for $D \geq 7$, we get that inverting all primes up to $\sqrt{D}$ the field is S-Euclidean. ■

Special Thanks

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