Distance sets in finite fields

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Euclidean Motivation

For $A \subseteq \mathbf{R}^2$, define the *distance set*:

$$\Delta(A) := \{|x - y| : x, y \in A\}$$

Goal: quantify "large sets must determine many distances"

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If $|A| = N$, *then* $|\Delta(A)| \gtrsim N/\log(N)$.

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Theorem (Wolff, 1999)

If $\dim(A) > 4/3$*, then* $m(\Delta(A)) > 0$*.*

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A Finite Field Model

$\mathbf{F}_q$: your favorite finite field with q large and odd

d: a small dimension

For $v, w \in \mathbf{F}_q^d$, define:

$$v \cdot w := \sum_{j=1}^{d} v_j w_j$$

$$|v|^2 := v \cdot v$$

For $A \subseteq \mathbf{F}_q^d$, define the *distance set*:

$$\Delta(A) := \{|x - y|^2 : x, y \in A\} \subseteq \mathbf{F}_q$$

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(if you prefer, rephrase in terms of quadratic forms)

Determining All Distances

Theorem (Iosevich, Rudnev 2007)

If $A \subseteq \mathbf{F}_q^d$ with $|A| > 2q^{\frac{d+1}{2}}$, then $\Delta(A) = \mathbf{F}_q$.

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compare with:

Theorem (Mattila, Sjölin 1999)

If $A \subseteq \mathbf{R}^d$ *with* $\dim(A) > \frac{d+1}{2}$, *then* $\Delta(A) \supseteq$ *an open interval.*

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Can we achieve the same conclusion with $|A| > 100q^{\frac{d}{2}+\frac{49}{100}}$?

(if d is odd, no)

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If $A \subseteq \mathbf{R}^d$ *with* $\dim(A) > \frac{d+1}{2}$, *then* $\Delta(A) \supseteq$ *an open interval.*

Can we achieve the same conclusion with $\dim(A) > \frac{d}{2} + \frac{49}{100}$?

(open in all dimensions $d \geq 2$)

Euclidean Distance Graphs

For $A \subseteq \mathbf{R}^d$, $\lambda \in \mathbf{R}$, define the λ-*distance graph* $\mathcal{G}_\lambda(A)$ by:

vertices : points of A

edges : connect $x, y \in A$ exactly when $|x - y| = \lambda$

Goal: quantify "large sets determine complex distance graphs"

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Theorem (Bennett, Iosevich, Taylor 2016)

If $A \subseteq \mathbf{R}^d$ with $\dim(A) > \frac{d+1}{2}$, then for every $k \in \mathbf{N}$, there exists an interval $(a, b) \subseteq \mathbf{R}$ such that for all $\lambda \in (a, b)$, $\mathcal{G}_\lambda(A)$ contains a path of length k.

nontrivial: $d \geq 2$

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Theorem (Greenleaf, Iosevich, Pramanik 2017)

If $A \subseteq \mathbf{R}^d$ with $\dim(A) > \frac{d+3}{2}$, then for every $k \in \mathbf{N}$, there exists an interval $(a, b) \subseteq \mathbf{R}$ such that for all $\lambda \in (a, b)$, $\mathcal{G}_\lambda(A)$ contains a cycle of length $2k$.

nontrivial: $d \geq 4$

Finite Field Distance Graphs

For $A \subseteq \mathbf{F}_q^d$, $\lambda \in \mathbf{F}_q$, define the λ-*distance graph* $\mathcal{G}_\lambda(A)$ by:

$$\begin{aligned} \text{vertices} &: \text{points of } A \\ \text{edges} &: \text{connect } x, y \in A \text{ exactly when } |x - y|^2 = \lambda \end{aligned}$$

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Goal: quantify “large sets determine complex distance graphs”

Theorem (Vinh 2012)

If $A \subseteq \mathbf{F}_q^d$, $|A| > C_n q^{\frac{d-3}{2}+n}$, then for every $\lambda \in \mathbf{F}_q^*$, $\mathcal{G}_\lambda(A)$ contains a complete graph on n vertices.

nontrivial: $d \geq 2n - 2$

Dimension Improvements

Theorem (P. 2017)

If $A \subseteq \mathbf{F}_q^d$, $|A| > C_n q^{d - \frac{n-1-d}{n}}$, then for every $\lambda \in \mathbf{F}_q^*$, $\mathcal{G}_\lambda(A)$ contains a complete graph on n vertices, provided K_n embeds in $\mathcal{G}_\lambda(\mathbf{F}_q^{n-1})$ with affine dimension $n-1$.

nontrivial: $d \geq n$

Embedding condition is somewhat natural: for instance, equilaterial triangles appear in $\mathbf{F}_p^2 \Leftrightarrow p \equiv 1, 3, 11 \mod 12$

Dimension Improvements

Theorem (Iosevich, P. 2017)

If $A \subseteq \mathbf{F}_q^d$ with $|A| > C_n q^{\frac{d-1}{2}+t}$, then for every $\lambda \in \mathbf{F}_q^*$, $\mathcal{G}_\lambda(A)$ contains all graphs on n vertices with maximum degree t.

nontrivial: $d \geq 2t$

Does a comparable Euclidean result hold?

(e.g. open for odd cycles)

Spherical Measures

For $\lambda \in \mathbf{F}_q^*$, $x \in \mathbf{F}_q^d$, define:

$$\sigma_\lambda(x) := \begin{cases} q & \text{if } |x|^2 = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

L^1 normalized in the sense:

$$\frac{1}{q^d} \sum_{x \in \mathbf{F}_q^d} \sigma_\lambda(x) = 1 + o(1)$$

where $o(1) \to 0$ as $q \to \infty$ provided $d \geq 2$.

In other words, all spheres are of size $\approx q^{d-1}$.

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$$\sigma_\lambda(x) := \begin{cases} q & \text{if } |x|^2 = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Exhibits Fourier decay in the sense:

$$\widehat{\sigma}_\lambda(\xi) = \begin{cases} 1 + o(1) & \text{if } \xi = 0 \\ o(1) & \text{if } \xi \neq 0 \end{cases}$$

where $o(1) \to 0$ as $q \to \infty$ provided $d \geq 2$.

Functional Iosevich-Rudnev

Theorem

For any $f, g : \mathbf{F}_q^d \to \mathbf{C}$, $\lambda \in \mathbf{F}_q^*$,

$$\frac{1}{q^{2d}} \sum_{x,y \in \mathbf{F}_q^d} f(x)g(y)\sigma_\lambda(x-y) = \frac{1}{q^{2d}} \sum_{x,y \in \mathbf{F}_q^d} f(x)g(y) + (error)$$

where $(error) = o(1)$ *if* $||f||_2||g||_2$ *is small with respect to q.*

$f = g = 1_A$ leads to original Iosevich-Rudnev distance theorem

Back to Distance Graphs

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Idea: edge deletion induction via functional Iosevich-Rudnev

Back to Distance Graphs

Theorem (P. 2017)

If $A \subseteq \mathbf{F}_q^d$, $|A| > C_n q^{d - \frac{n-1-d}{n}}$, *then for every* $\lambda \in \mathbf{F}_q^*$, $\mathcal{G}_\lambda(A)$ *contains a complete graph on n vertices, provided* K_n *embeds in* $\mathcal{G}_\lambda(\mathbf{F}_q^{n-1})$ *with affine dimension* $n-1$.

Idea: vertex deletion induction with refined spherical measures; Fourier decay of intersections of spheres

Thanks for your attention.