Numerical Experiments with Noncototients

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Noncototients (Guy B36)

Let $s_\varphi(n) = n - \varphi(n)$, where φ is Euler's totient function.

$m \in \mathbb{Z}^+$ is a *noncototient* if $m \neq s_\varphi(n)$ for any $n \in \mathbb{Z}$

Assuming every even $n = p + q > 6$, all noncototients are even

Erdős, Sierpínski (1956): are there infinitely many?

- Browkin, Schinzel (1995): yes — $2^k \cdot 509203$

Asymptotic Density?

Erdős (1974): Positive asymptotic density?

- Pomerance, Yang (2013): enumeration of even noncototients to 10^8
- Pollack, Pomerance (2016): enumeration to 10^{10}, conjectural density
  $$\Delta_\varphi := \lim_{x\to\infty} \frac{1}{\log x} \sum_{a\le x}^{*} \frac{1}{a} e^{-a/s_\varphi(a)} \approx 0.0908721526$$
  (sum over even a)

Our contribution:

- Extend enumeration to 10^{12}
- Slight improvement to Pomerance and Yang algorithm
- Investigate other distribution questions

The Algorithm of Pomerance and Yang

Enumerate all cototients using a sieve, based on (k odd integer):

- $s_\varphi(2k) = 2k - \varphi(k)$
- $s_\varphi(2^{j+1}k) = 2s_\varphi(2^j k)$

All even integers left are noncototients.

Improvement: enumerate noncototients directly, using that every even noncototient is twice an odd number or twice another noncototient

$O(n \log n)$ to compute a table of $\varphi(k)$ values, $O(n)$ for enumeration

- Constant improvement in enumeration stage (identifying noncototients directly instead of cototients)

Implementation and Running Time

Noncototients to 4×10^{11} :

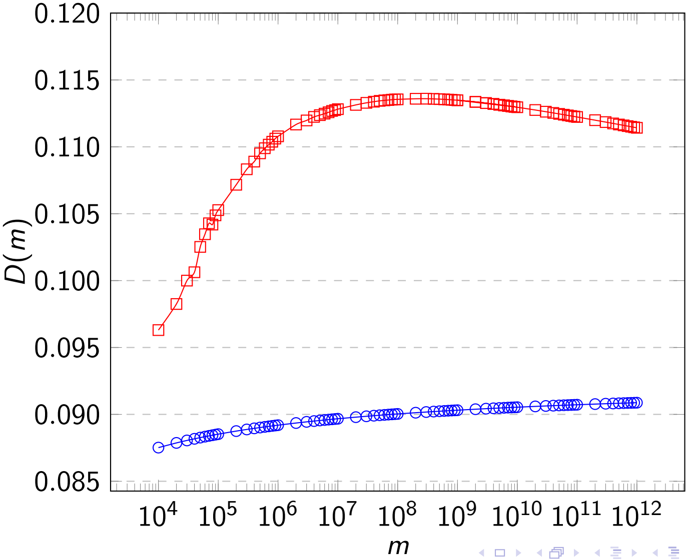
- 2.2 GHz, 2 TB RAM
- 10.78 hours to compute $\varphi(k)$ values
- Original algorithm: 4.24 hours for enumeration
- New version: 2.89 hours

Noncototients to 10^{12} (using segmented enumeration of $\varphi(k)$ values)

- 2.27 GHz, 256 GB RAM
- about a week in total

1.4 TB of data

Density of Even Noncototients $\leq m$



Noncototient Classes and Chains

A noncototient *class* is the noncototients generated by repeatedly multiplying an odd integer by two until the result is no longer a noncototient.

- 10 is the sole member of its class (base 5, 20 is a cototient)
- 26 is the first in a class of length two (26, 52)
- 1018406 is the first term in an infinite class (Browkin, Schinzel)

A noncototient *chain* is a sequence of consecutive even noncototients.

- 10 is a class of length one; $50, 52$ is a class of length two (or *pair*)

Do classes and chains of every length exist?

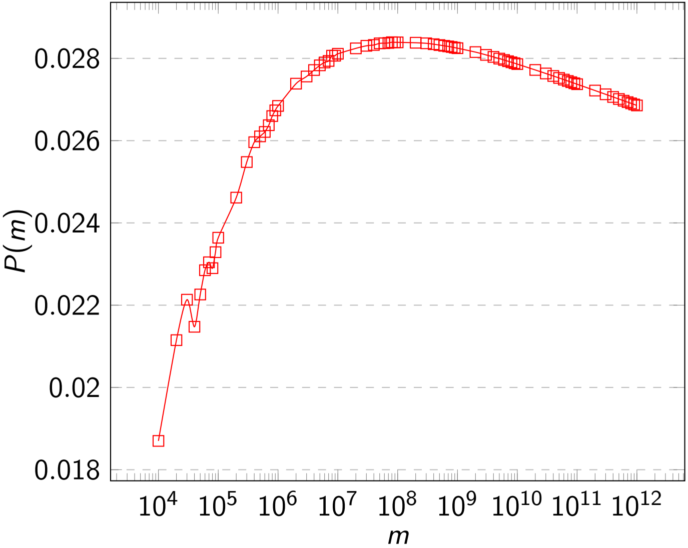
Noncototient Classes

Class Length (l)	First Appearance	Number of Classes
Unknown	1286	55587470339
1	10	2144082157
2	26	1115413259
3	58	451275610
4	134	235917405
5	634	86147965
⋮	⋮	⋮
20	3074	537
21	1522	154
22	1886	145
23	11586	21
24	2870	21
25	2626	3
26	2486	4
27	0	0
28	3554	1

Noncototient Chains

Chain Length (l)	First Appearance	Number of Chains
1	10	64217003593
2	50	15408230556
3	532	3755744247
4	2314	896000615
5	4628	219769433
6	22578	52947013
7	115024	12660551
8	221960	3014671
9	478302	703130
10	3340304	163297
11	22527850	39589
12	117335136	9497
13	1118736102	2316
14	1564578508	524
15	6121287812	121
16	7515991946	38
17	470344908044	2
18	300899994422	1
19	0	0
20	234063318774	1

Noncototient Pairs — Density?



Open Problems

Prove the Pollack and Pomerance density conjecture?

Are there noncototient classes of every integer length $l \geq 1$?

Are there noncototient chains of every integer length $l \geq 1$?

Are there infinitely many noncototient pairs?

Is there a positive asymptotic density of noncototient pairs?