Consecutive integers divisible by a power of their largest prime factor

West Coast Number Theory Conference

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1\,294\,299 &= 3^4 \cdot 19 \cdot 29^2 \\
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$$E_{k,\ell} := \{n \in \mathbb{N} : P(n+i)^\ell \mid n+i \text{ for each } i = 0, 1, \ldots, k-1\}.$$

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so that $1\,294\,298 \in E_{3,2}$

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- $632\,127\,050\,601\,113\,666\,430 \in E_{2,4}$

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  In 2017, de la Bretèche and Drappeau replace 25/24 by 1.33

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Consider the polynomials

$$\begin{aligned} f(x) &= x^2(2x^3+5x^2-5) \\ f(x)+1 &= (x^2+x-1)^2(2x+1) \\ f(x)+2 &= (x+1)^2(2x^3+x^2-4x+2) \end{aligned}$$

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With $x = 3802$, we find

$$\begin{aligned} 1\,589\,922\,788\,612\,140\,124 &= 2^2 \cdot 59 \cdot 61 \cdot 71 \cdot 239 \cdot 1801 \cdot 1901^2 \\ 1\,589\,922\,788\,612\,140\,125 &= 3^2 \cdot 5^3 \cdot 11^2 \cdot 13^2 \cdot 151^2 \cdot 1741^2 \\ 1\,589\,922\,788\,612\,140\,126 &= 2 \cdot 103 \cdot 701 \cdot 809 \cdot 941 \cdot 3803^2 \end{aligned}$$

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With $x = 5087$, we find the somewhat larger 23 digit numbers

$$\begin{aligned} 69\,315\,509\,064\,481\,032\,011\,329 &= 2^6 \cdot 3^7 \cdot 53 \cdot 53 \cdot 2543 \cdot 1916857^2 \\ 69\,315\,509\,064\,481\,032\,011\,330 &= 11 \cdot 71 \cdot 769 \cdot 1163 \cdot 1321 \cdot 2903 \cdot 5087^2 \\ 69\,315\,509\,064\,481\,032\,011\,331 &= 2 \cdot 5 \cdot 7^2 \cdot 17 \cdot 29^2 \cdot 167^2 \cdot 181 \cdot 44273^2 \end{aligned}$$

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Since $x = mP(m)$ is such that $P(x)^2 \mid x$ for each $m \geq 2$, we substitute $x = mP(m)$ in the above system and letting m run up to 30 000 000, we find 4 473 elements of $E_{3,2}$

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$$\begin{aligned} g(x) - 1 &= (8x-1)^3(24x^2+9x+1), \\ g(x) &= x(12288x^4-640x^2+15), \\ g(x) + 1 &= (8x+1)^3(24x^2-9x+1) \end{aligned}$$

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$$\begin{aligned} g(x) - 1 &= (32x+1)^4(10240x^3-1280x^2+58x-1), \\ g(x) &= 2x(5368709120x^6-22020096x^4+35840x^2-35), \\ g(x) + 1 &= (32x-1)^4(10240x^3+1280x^2+58x+1). \end{aligned}$$

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Can we find 3 consecutive polynomials of the type

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implying that

$$g(x) = \int f_0(x)\,dx$$

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This gives $g(x) = a\left(\frac{16x^7}{7} - \frac{8x^5}{5} + \frac{x^3}{3}\right)$.

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Choosing $a=105$, we find

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Let p be a prime such that $6p-1$ and $6p+1$ are primes. Then, with $x = 3p = 3 \cdot 84\,530\,423\,897$, we find

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Let p be a prime such that $6p-1$ and $6p+1$ are primes. Then, with $x = 3p = 3 \cdot 84\,530\,423\,897$, we find the 83-digit number n satisfying

$$\begin{aligned} n-1 &= 2^4 \cdot 379 \cdot 607 \cdot 719 \cdot 5903 \cdot 150095257303 \cdot 166390767539\\ &\quad \cdot 317989448093 \cdot 507182543381^3\\ n &= 3^3 \cdot 19 \cdot 271 \cdot 200357 \cdot 681259 \cdot 1092103 \cdot 7783463\\ &\quad \cdot 3941566193 \cdot 42150574033 \cdot 84530423897^3\\ n+1 &= 2 \cdot 13 \cdot 101 \cdot 58049 \cdot 258067 \cdot 290317 \cdot 28931491\\ &\quad \cdot 11614182259 \cdot 32329918001 \cdot 507182543383^3 \end{aligned}$$

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In 2014, Peter Burcsi and Gabor Gévay found the 77-digit integer n which satisfies

$$\begin{aligned}
n-1 &= 2^7 \cdot 53 \cdot 4253 \cdot 27631 \cdot 27953 \cdot 1546327 \cdot 2535271 \\
&\quad \cdot 17603683 \cdot 1472289739 \cdot 16476952799^3 \\
n &= 3^6 \cdot 19 \cdot 37 \cdot 787 \cdot 711163 \cdot 2181919 \cdot 137861107 \\
&\quad \cdot 318818473 \cdot 937617607 \cdot 7323090133^3 \\
n+1 &= 2 \cdot 12899 \cdot 133451 \cdot 421607 \cdot 2198029 \cdot 8046041 \\
&\quad \cdot 19854409 \cdot 555329197 \cdot 32953905599^3,
\end{aligned}$$

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$$\begin{aligned} g(x) &= x^6(252x^5 - 1386x^4 + 3080x^3 - 3465x^2 + 1980x - 462),\\ g(x)+1 &= (x-1)^6(252x^5 + 126x^4 + 56x^3 + 21x^2 + 6x + 1). \end{aligned}$$

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Choosing $x = 86\,459\,129\,774$, we find a 123-digit integer n with $P(n)^6 = 43\,229\,564\,887^6 \mid n$ and $P(n+1)^6 = 86\,459\,129\,773^6 \mid n+1$.

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such that each $L_i(x)$ is divisible by the ℓ-th power of some linear polynomial.

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Example $(k = 4,\ \ell = 2)$

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Example $(k = 4,\ \ell = 2)$

$$\begin{aligned}
f(x) &= 4x^2(184896x^5 + 292320x^4 + 172500x^3 + 46500x^2 + 5501x + 195) \\
f(x)+1 &= (2x+1)^2(184896x^5 + 107424x^4 + 18852x^3 + 792x^2 - 4x + 1) \\
f(x)+2 &= 2(4x+1)^2(23112x^5 + 24984x^4 + 7626x^3 + 438x^2 - 8x + 1) \\
f(x)+3 &= (6x+1)^2(20544x^5 + 25632x^4 + 10052x^3 + 1104x^2 - 36x + 3)
\end{aligned}$$

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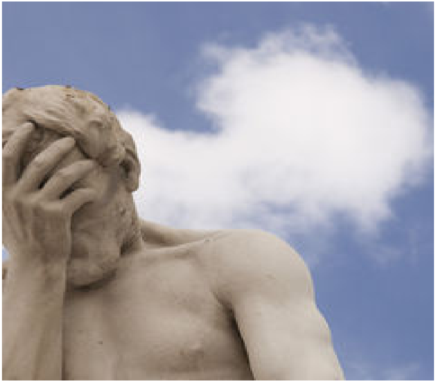
implying that the smallest element of $E(3,3)$ has around 82 digits and that the smallest element of $E(4,2)$ has around 71 digits

Food for thought. . .



Thank you !

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