Computing Periods of Modular Curves with Vector-Valued Modular Forms

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$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) \in \Gamma \times \mathbb{H}^* \mapsto \frac{a\tau+b}{c\tau+d} \in \mathbb{H}^*.$$

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The **genus** of G is the genus of $G\backslash\mathbb{H}^*$, i.e. the number of "holes" in the underlying surface.

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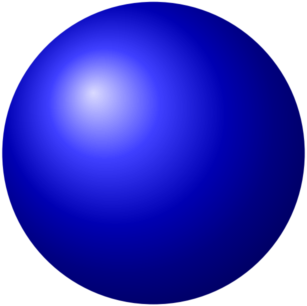

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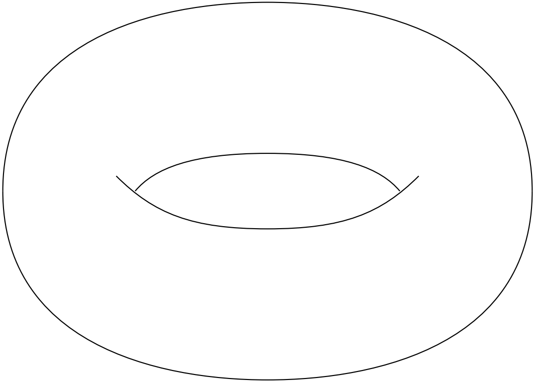

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These functions form a $\mathbb{C}$-linear space $S_2(G)$, whose dimension is the genus of G.

Homology of $G\backslash\mathbb{H}^*$

Fix the base point $i\infty$ for $H_1(G\backslash\mathbb{H}^*, \mathbb{Z})$. Each "hole" in $G\backslash\mathbb{H}^*$ has an "A-cycle" and a "B-cycle" associated to it, for a total of $2g$ independent closed paths where $g \geq 1$ is the genus of G:

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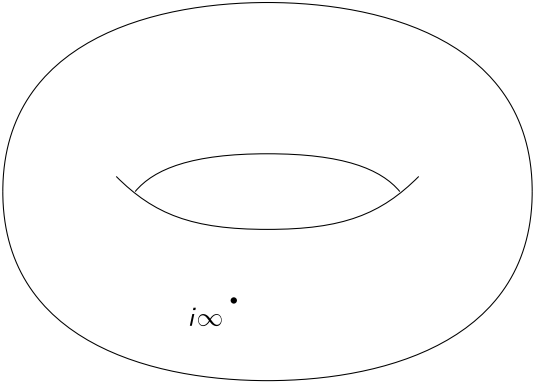

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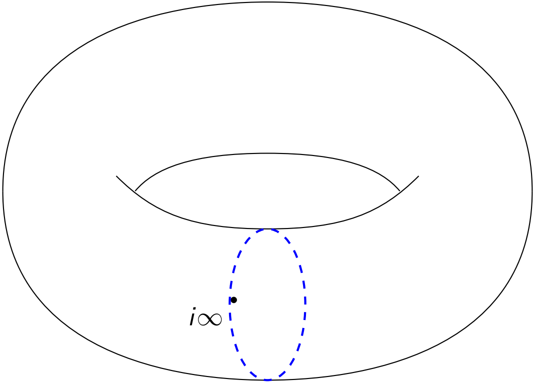

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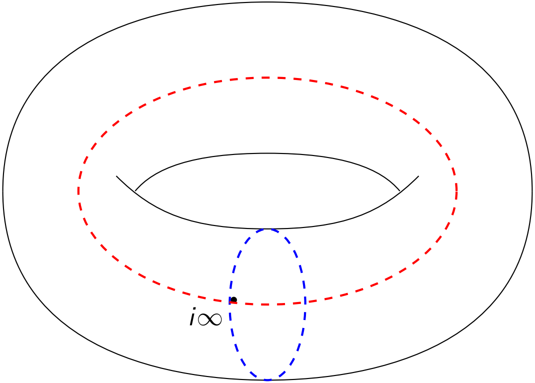

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These numbers span a full rank lattice Λ in $\mathbb{C}^g$ (i.e. a free $\mathbb{Z}$-module of rank $2g$), and $\mathbb{C}^g/\Lambda$ is an **abelian variety** called the **Jacobian** of the modular curve $G\backslash\mathbb{H}^*$.

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Consequently, the vector $F = (f_1, \cdots, f_g)^t$ is a **weight two vector-valued cusp form** for a representation $\rho_0 : \Gamma \to \mathrm{GL}_g(\mathbb{C})$.

Thus for each $\gamma \in \Gamma$ we have $F|_2\gamma = \rho_0(\gamma)F$.

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Since

$$U(\gamma(i\infty)) = \rho(\gamma)U(i\infty) = \rho(\gamma)(0, \cdots, 0, 1)^t = (\Omega(\gamma), 1)^t$$

we see that the vectors $\Omega(\gamma_k)$ span the period lattice Λ for $G\backslash\mathbb{H}^*$, where $\{\gamma_k\}_{k=1}^{2g}$ gives the homology group for $G\backslash\mathbb{H}^*$ as above.

Examples

There is an infinite family of genus one subgroups

$$\Gamma'' < G_{p,m,d} \leq \Gamma' = G_{1,0,1}$$

that are normal in Γ, where $p, m, d \in \mathbb{N}$ and $m^2 + m + 1 \equiv 0 \pmod{d}$.

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Thus each such G defines an elliptic curve with complex multiplication.

CM Jacobians

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This is ongoing work with Luca Candelori.

Thanks very much!