Geometric representations of triangular squares

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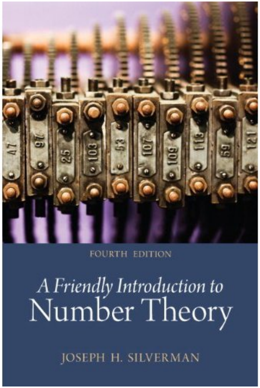
The Book and Author



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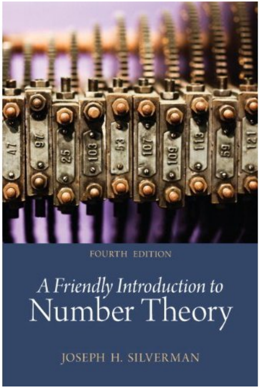

"An introductory undergraduate text designed to entice non-math majors into learning some mathematics, while at the same time teaching them how to think mathematically."

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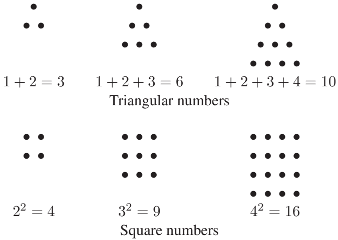

Figure 1.1: Numbers That Form Interesting Shapes

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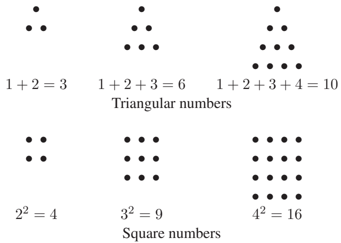

Figure 1.1: Numbers That Form Interesting Shapes

What do the problems look like?

Problem 1

Problem

The first two numbers that are both squares and triangles are 1 and 36. Find the next one and, if possible, the one after that. Can you figure out an efficient way to find triangular-square numbers? Do you think that there are infinitely many?

How hard could it be?

Square numbers: **1**, 4, 9, 16, 25, **36**, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089, 1156, **1225**, 1296, 1369, 1444, 1521, 1600, 1681, 1764, 1849, 1936, 2025, 2116, 2209, 2304, 2401, 2500, 2601, 2704, 2809, 2916, 3025, 3136, 3249, 3364, 3481, 3600, 3721,. . .

Triangular numbers: **1**, 3, 6, 10, 15, 21, 28, **36**, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, **1225**, 1275, 1326, 1378, 1431, 1485, 1540, 1596, 1653, 1711, 1770, 1830, 1891, 1953, 2016, 2080, 2145, 2211, 2278, 2346, 2415, 2485, 2556, 2628, 2701, 2775, 2850, 2926, 3003, 3081, 3160, 3240, 3321, 3403, 3486, 3570, 3655, 3741,. . .

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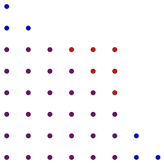
- Student: No.
- Instructor: Yes.
- Mits wearing student's hat: Maybe, if you can solve it using only material in chapter 1.

A first attempt

Manipulating dots, we can overlap equal square and triangle:



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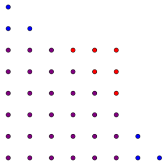

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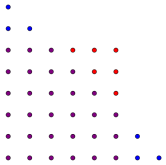

$$S_6 = T_8 \iff 2T_2 = T_3$$

So whenever a triangle equals a square, we also have $2T_m = T_n$ for some m and n, and vice versa.

Let's run with this. . .

Even triangular numbers: **6**, 10, 28, 36, 66, 78, 120, 136, 190, **210**, 276, 300, 378, 406, 496, 528, 630, 666, 780, 820, 946, 990, 1128, 1176, 1326, 1378, 1540, 1596, 1770, 1830, 2016, 2080, 2278, 2346, 2556, 2628, 2850, 2926, 3160, 3240, 3486, 3570, 3828, 3916, 4186, 4278, 4560, 4656, 4950, 5050, 5356, 5460, 5778, 5886, 6216, 6328, 6670, 6786, **7140**, 7260, 7626, 7750,. . .

Double-triangular numbers: 2, **6**, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, **210**, 240, 272, 306, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 992, 1056, 1122, 1190, 1260, 1332, 1406, 1482, 1560, 1640, 1722, 1806, 1892, 1980, 2070, 2162, 2256, 2352, 2450, 2550, 2652, 2756, 2862, 2970, 3080, 3192, 3306, 3422, 3540, 3660, 3782, 3906, 4032, 4160, 4290, 4422, 4556, 4692, 4830, 4970, 5112, 5256, 5402, 5550, 5700, 5852, 6006, 6162, 6320, 6480, 6642, 6806, 6972, **7140**, 7310, 7482, 7656, 7832,. . .

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where $m = 84, n = 119.$

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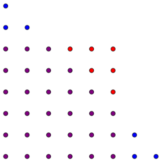

Another triangular square!

$$7140 = 2T_m = T_n$$

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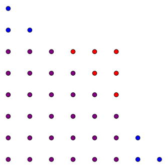
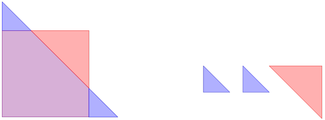


This means $S_{84+119+1} = 204^2 = 41616$ is a triangular-square number.

A related idea

We just started with an example of a triangular-square number and "reduced" it to an example of a (smaller) triangular-double triangular number.



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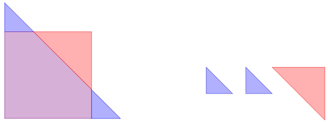

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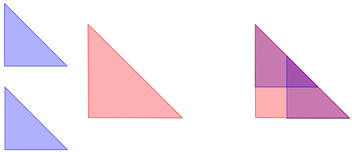

Combined

$$S_{m_1} = T_{n_1} \Rightarrow 2T_a = T_b \Rightarrow S_{m_2} = T_{n_2}$$

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$$\begin{aligned} m_{k+1} &= 3m_k + 2n_k + 1 \\ n_{k+1} &= 4m_k + 3n_k + 1 \end{aligned}$$

$$\begin{aligned} m_1 = 1, n_1 = 1, &\quad S_1 = T_1 = 1 \\ m_2 = 6, n_2 = 8, &\quad S_6 = T_8 = 36 \\ m_3 = 35, n_3 = 49, &\quad S_{35} = T_{49} = 1225 \\ m_4 = 204, n_4 = 288, &\quad S_{204} = T_{288} = 41616 \\ m_5 = 1189, n_5 = 1681, &\quad S_{1189} = T_{1681} = 1413721 \end{aligned}$$

Is there any more to this?

How could we generalize the $S_m = T_n$ problem?

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Thus, it suffices to study

$$kS_m = T_n.$$

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$$T_k S_m = T_n$$

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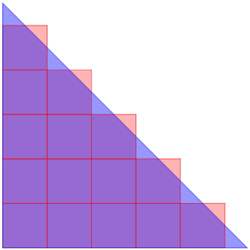

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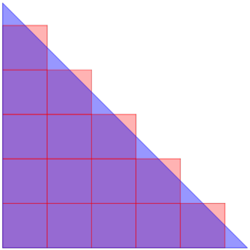

$$(k+1)T_a = kT_b$$

An alternative derivation

$$T_5 S = T$$

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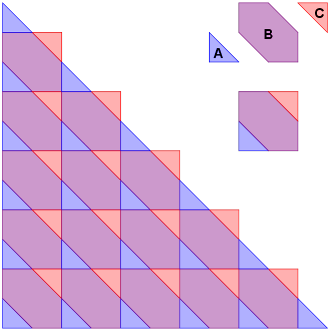
$$S = A + B + C$$

$$T = T_6 A + T_5 B + T_4 C$$



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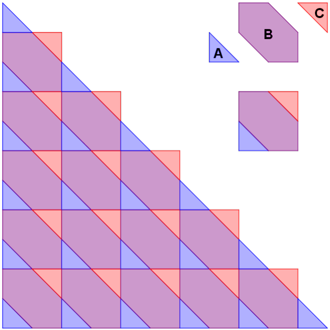

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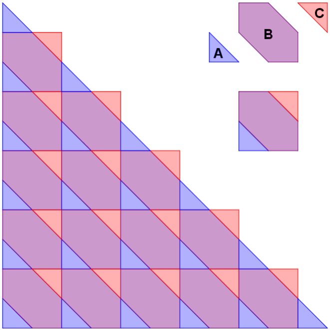

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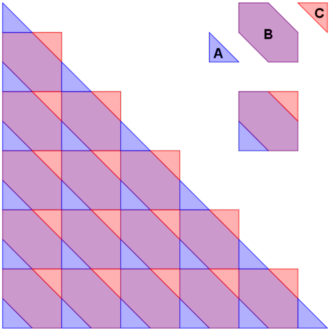

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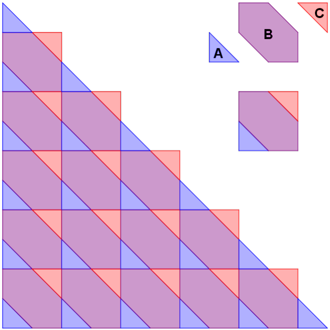

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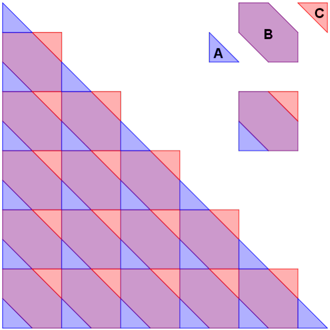

$$T_k S = T \Leftrightarrow kC = (k+1)A$$

Existence

$$(k+1)T_{2k} = kT_{2k+1}$$

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Can we use this identity as a "seed" to generate infinitely many $(k+1)T_a = kT_b$?

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We will need the related identities

$$\begin{aligned}(k+1)T_{2k+1} - kT_{2k+2} &= k+1\\ (k+1)T_{2k} - kT_{2k+1} &= 0\\ (k+1)T_{2k-1} - kT_{2k} &= -k\end{aligned}$$

Ascent

Let $A_1 = T_{2k}, C_1 = T_{2k+1}$, then the identities are

$$\begin{cases} (k+1)A_1^+ - kC_1^+ = k+1 \\ (k+1)A_1 - kC_1 = 0 \\ (k+1)A_1^- - kC_1^- = -k \end{cases}$$

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Given $(k+1)A = kC$, and implied B, define

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Observe that both A' and C' are triangles (larger than A and C).

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Claim: $(k+1)A' = kC$.

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$$(k+1)A = kC$$

$$\begin{aligned} (k+1)A' &= (k+1)A_1^+ A + (k+1)A_1 B + (k+1)A_1^- C \\ kC' &= kC_1^+ A + kC_1 B + kC_1^- C \end{aligned}$$

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Theorem

For each k, there are infinitely many S, T satisfying $T_k S = T$.

What's next?

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- What about $kS = T$, general k?

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- How does this relate to...?