Shifted Convolution L-Series Values of Elliptic Curves

Nitya Mani (joint with Asra Ali)

December 18, 2017

Preliminaries

Modular Forms for $\Gamma_0(N)$

Modular Forms for $\Gamma_0(N)$

Definition

The congruence subgroup $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ is

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}.$$

Definition

A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight k for $\Gamma_0(N)$ if

- ☐ f is holomorphic,
- ☐ $f(z) = (cz + d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ for all $\gamma \in \Gamma_0(N)$,
- ☐ f is holomorphic at the **cusps**: the orbits of $\mathbb{P}_1(\mathbb{Q})$ under $\Gamma_0(N)$.

Elliptic Curves are Modular

Elliptic Curves are Modular

- Consider $z = x + iy \in \mathcal{H}$ where $\mathcal{H} = \{z \in \mathbb{C} | y > 0\}$
- Any elliptic curve $E/\mathbb{Q}$ has associated weight 2 newform

$$f_E(z) = \sum_{n=1}^{\infty} a_E(n) q^n \qquad a_E(1) = 1,\ q = e^{2\pi i z}$$

Elliptic Curves are Modular

- Consider $z = x + iy \in \mathcal{H}$ where $\mathcal{H} = \{z \in \mathbb{C} | y > 0\}$
- Any elliptic curve $E/\mathbb{Q}$ has associated weight 2 newform

$$f_E(z) = \sum_{n=1}^{\infty} a_E(n) q^n \qquad a_E(1) = 1,\ q = e^{2\pi i z}$$

- f_E a modular form for $\Gamma_0(N_E)$, N_E conductor of E

Elliptic Curves are Modular

- Consider $z = x + iy \in \mathcal{H}$ where $\mathcal{H} = \{z \in \mathbb{C} | y > 0\}$
- Any elliptic curve $E/\mathbb{Q}$ has associated weight 2 newform

$$f_E(z) = \sum_{n=1}^{\infty} a_E(n) q^n \qquad a_E(1) = 1,\ q = e^{2\pi i z}$$

- f_E a modular form for $\Gamma_0(N_E)$, N_E conductor of E
- Modularity Theorem + Eichler-Shimura Theory (Taylor, Wiles, Conrad, et al.)

Elliptic Curves are Modular

- Consider $z = x + iy \in \mathcal{H}$ where $\mathcal{H} = \{z \in \mathbb{C} | y > 0\}$
- Any elliptic curve $E/\mathbb{Q}$ has associated weight 2 newform

$$f_E(z) = \sum_{n=1}^{\infty} a_E(n) q^n \qquad a_E(1) = 1,\ q = e^{2\pi i z}$$

- f_E a modular form for $\Gamma_0(N_E)$, N_E conductor of E
- Modularity Theorem + Eichler-Shimura Theory (Taylor, Wiles, Conrad, et al.)
- p prime: $a_E(p) = p + 1 - \#E(\mathbb{F}_p)$

Shifted Convolution L-series

- Modular form: $f_E(z) = \sum_{n=1}^{\infty} a_E(n) q^n$
- Dirichlet L-series:

$$L(E, s) = L(f_E, s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}$$

Shifted Convolution L-series

- Modular form: $f_E(z) = \sum_{n=1}^{\infty} a_E(n) q^n$
- Dirichlet L-series:

$$L(E, s) = L(f_E, s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}$$

- Shifted convolution L-series:

$$D_{f_E}(h; s) = \sum_{n=1}^{\infty} a_E(n+h)\overline{a_E(n)} \left(\frac{1}{(n+h)^s} - \frac{1}{n^s} \right)$$

Shifted Convolution L-series

- Modular form: $f_E(z) = \sum_{n=1}^{\infty} a_E(n) q^n$
- Dirichlet L-series:

$$L(E,s) = L(f_E,s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}$$

- Shifted convolution L-series:

$$D_{f_E}(h;s) = \sum_{n=1}^{\infty} a_E(n+h)\overline{a_E(n)} \left(\frac{1}{(n+h)^s} - \frac{1}{n^s} \right)$$

- Generating function ($s=1$):

$$\mathbb{L}_{f_E}(z) = \sum_{h=1}^{\infty} D_{f_E}(h;1) q^h$$

Previous Work

- Shifted convolution L-series defined by Hoffstein, Hulse, Reznikov (generalized Rankin-Selberg convolutions)

Previous Work

- Shifted convolution L-series defined by Hoffstein, Hulse, Reznikov (generalized Rankin-Selberg convolutions)
- Values are essentially coefficients of mixed mock modular forms; recent work on p-adic properties and asymptotic behavior (Beckwith, Bringmann, Mertens, Ono)

Goals

Goals

- ☐ Given $E/\mathbb{Q}$, we have weight 2 newform:

$$f_E = \sum_{n=1}^{\infty} a_E(n) q^n, \quad a_E(1) = 1$$

- ☐ Understand variants of $L(f_E, s)$ at $s = 1$

Goals

- ☐ Given $E/\mathbb{Q}$, we have weight 2 newform:

$$f_E = \sum_{n=1}^{\infty} a_E(n)q^n, \quad a_E(1) = 1$$

- ☐ Understand variants of $L(f_E, s)$ at $s = 1$
- ☐ Closed form for $D_{f_E}(h; 1)$ values for some $E/\mathbb{Q}$

Weierstrass ζ-function

Weierstrass ζ-function

- $E \simeq \mathbb{C}/\Lambda_E$, where Λ_E a 2-dimensional complex lattice

Weierstrass ζ-function

- $E \simeq \mathbb{C}/\Lambda_E$, where Λ_E a 2-dimensional complex lattice
- Weierstrass ζ-function:

$$\zeta(\Lambda_E; z) = \frac{1}{z} + \sum_{w \in \Lambda_E \setminus \{0\}} \left(\frac{1}{z - w} + \frac{1}{w} + \frac{z}{w^2} \right)$$

Weierstrass ζ-function

- $E \simeq \mathbb{C}/\Lambda_E$, where Λ_E a 2-dimensional complex lattice
- Weierstrass ζ-function:

$$\zeta(\Lambda_E; z) = \frac{1}{z} + \sum_{w \in \Lambda_E \setminus \{0\}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right)$$

- $\frac{d}{dz}\zeta(\Lambda_E; z) = -\wp(\Lambda_E; z)$
- (Eisenstein) Lattice-invariant function $\mathfrak{Z}_E(z)$

$$\mathfrak{Z}_E(z) = \zeta(\Lambda_E; z) - S(\Lambda_E)z - \frac{\pi}{\mathrm{vol}(\Lambda_E)}\overline{z}$$

$$S(\Lambda_E) = \lim_{s \to 0} \sum_{w \in \Lambda_E \setminus \{0\}} \frac{1}{w^2 |w|^{2s}}$$

Weierstrass Mock Modular Form

Weierstrass Mock Modular Form

- Eichler integral of associated newform f_E:

$$\mathcal{E}_{f_E}(z) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n$$

Weierstrass Mock Modular Form

- Eichler integral of associated newform f_E:

$$\mathcal{E}_{f_E}(z) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n$$

- Harmonic Maaß form $\widehat{\mathfrak{Z}}_E(z)$:

$$\widehat{\mathfrak{Z}}_E(z) = \mathfrak{Z}_E(\mathcal{E}_{f_E}(z))$$

Weierstrass Mock Modular Form

- Eichler integral of associated newform f_E:

$$\mathcal{E}_{f_E}(z) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n$$

- Harmonic Maaß form $\widehat{\mathfrak{Z}}_E(z)$:

$$\widehat{\mathfrak{Z}}_E(z) = \mathfrak{Z}_E(\mathcal{E}_{f_E}(z))$$

- Can split $\widehat{\mathfrak{Z}}_E(z)$: $\widehat{\mathfrak{Z}}_E(z) = \widehat{\mathfrak{Z}}_E^+(z) + \widehat{\mathfrak{Z}}_E^-(z)$

Weierstrass Mock Modular Form

- Eichler integral of associated newform f_E:

$$\mathcal{E}_{f_E}(z) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n$$

- Harmonic Maaß form $\widehat{\mathfrak{Z}}_E(z)$:

$$\widehat{\mathfrak{Z}}_E(z) = \mathfrak{Z}_E(\mathcal{E}_{f_E}(z))$$

- Can split $\widehat{\mathfrak{Z}}_E(z)$: $\widehat{\mathfrak{Z}}_E(z) = \widehat{\mathfrak{Z}}_E^+(z) + \widehat{\mathfrak{Z}}_E^-(z)$
- If genus$(X_0(N)) = 1$, $\widehat{\mathfrak{Z}}_E^+(z)$ is holomorphic; called the **Weierstrass mock modular form**

$$\mathfrak{Z}_E^+(\mathcal{E}_{f_E}(z)) = \zeta(\Lambda_E; \mathcal{E}_{f_E}(z)) - S(\Lambda_E)\mathcal{E}_{f_E}(z)$$

Example ($N = 27$)

- ☐ $N = 27$ and $E_{27} : y^2 + y = x^3 - 7$

Example ($N = 27$)

- ☐ $N = 27$ and $E_{27} : y^2 + y = x^3 - 7$
- ☐ Weight 2 modular form associated with E_{27}:

$$f_{E_{27}} = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + O(q^{20}).$$

Example ($N = 27$)

- $N = 27$ and $E_{27} : y^2 + y = x^3 - 7$
- Weight 2 modular form associated with E_{27}:

$$f_{E_{27}} = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + O(q^{20}).$$

- Weight 0 mock modular form $\widehat{\mathfrak{Z}}^+_{E_{27}}(z)$ associated to $f_{E_{27}}$:

$$\widehat{\mathfrak{Z}}^+_{E_{27}}(z) = q^{-1} + \frac{1}{2}q^2 + \frac{1}{5}q^5 + \frac{3}{4}q^8 - \frac{6}{11}q^{11} - \frac{1}{2}q^{14} + O(q^{17})$$

Main Results

Elliptic Curves

- ☐ $E/\mathbb{Q}$ is a strong Weil elliptic curve
- ☐ If E has conductor N_E, then

$$\textbf{genus}(X_0(N_E)) = 1$$

Elliptic Curves

- $E/\mathbb{Q}$ is a strong Weil elliptic curve
- If E has conductor N_E, then

$$\textbf{genus}(X_0(N_E)) = 1$$

- Two situations:
  1. N_E is **squarefree**

$$N \in \{11, 14, 15, 17, 19, 21\}$$

Elliptic Curves

- $E/\mathbb{Q}$ is a strong Weil elliptic curve
- If E has conductor N_E, then

$$\textbf{genus}(X_0(N_E)) = 1$$

- Two situations:
  1. N_E is **squarefree**

     $$N \in \{11, 14, 15, 17, 19, 21\}$$

  2. E has **complex multiplication**

     $$N \in \{27, 32, 36, 49\}$$

Theorem 1

- E strong Weil curve, $\text{genus}(X_0(N)) = 1$, N **squarefree**
- Recall $\mathbb{L}_{f_E}(z) = \sum_{h=1}^{\infty} D_{f_E}(h;1)q^h$

Theorem 1

- E strong Weil curve, $\text{genus}(X_0(N)) = 1$, N **squarefree**
- Recall $\mathbb{L}_{f_E}(z) = \sum_{h=1}^{\infty} D_{f_E}(h;1)q^h$

Theorem (A-M '17)

$$\mathbb{L}_{f_E}(z;1) = \frac{\text{vol}(\Lambda_E)}{\pi}\left((f_E(z) \cdot \widehat{\mathfrak{Z}}_E^+(z)) - \alpha f_E(z) - \sum_i F_{N,2}^{\infty}(z) \right)$$

- $F_{N,2}^{\infty}(z)$ *is the Eisenstein series for* $\Gamma_0(N)$ *nonvanishing only at the cusp* ∞
- $\alpha = (f_E \cdot \widehat{\mathfrak{Z}}_E^+)[1] - \frac{\pi}{\text{vol}\,\Lambda_E} D_{f_E}(1;1) - F_{N,2}^{\infty}[1]$

Theorem 2

- E is a strong Weil elliptic curve with **complex multiplication** where $\text{genus}(X_0(N)) = 1$
- Recall $\mathbb{L}_{f_E}(z) = \sum_{h=1}^{\infty} D_{f_E}(h;1)q^h$
- $F_{N,2}^{\infty}$ is the Eisenstein series for $\Gamma_0(N)$ nonvanishing only at the cusp ∞

Theorem 2

- E is a strong Weil elliptic curve with **complex multiplication** where genus$(X_0(N)) = 1$
- Recall $\mathbb{L}_{f_E}(z) = \sum_{h=1}^{\infty} D_{f_E}(h;1)q^h$
- $F_{N,2}^{\infty}$ is the Eisenstein series for $\Gamma_0(N)$ nonvanishing only at the cusp ∞

Theorem (A-M '17)

With $\mathbb{L}_{f_E}(z)$ defined as above,

$$\mathbb{L}_{f_E}(z;1) = \frac{\mathrm{vol}(\Lambda_E)}{\pi}\left(\left(f_E(z) \cdot \widehat{\mathfrak{Z}}_E^{+}\right)(z) - F_{N,2}^{\infty}(z)\right).$$

Proof Idea: Holomorphic Projection

Motivation

- ☐ Consider $f : \mathcal{H} \to \mathbb{C}$ continuous, transforms like a modular form for $\Gamma_0(N)$ of weight $k \geq 2$, has moderate growth at cusps

Motivation

- Consider $f : \mathcal{H} \to \mathbb{C}$ continuous, transforms like a modular form for $\Gamma_0(N)$ of weight $k \geq 2$, has moderate growth at cusps
- Induces a linear functional based on the **Petersson Inner Product**

$$g \to \langle g, f \rangle$$

for $g \in S_k(\Gamma_0(N))$

Motivation

- Consider $f : \mathcal{H} \to \mathbb{C}$ continuous, transforms like a modular form for $\Gamma_0(N)$ of weight $k \geq 2$, has moderate growth at cusps
- Induces a linear functional based on the **Petersson Inner Product**

$$g \to \langle g, f \rangle$$

  for $g \in S_k(\Gamma_0(N))$
- $\implies$ There is some $\widetilde{f} \in S_k(\Gamma_0(N))$ such that $\langle \cdot, f \rangle = \langle \cdot, \widetilde{f} \rangle$

Motivation

- Consider $f : \mathcal{H} \to \mathbb{C}$ continuous, transforms like a modular form for $\Gamma_0(N)$ of weight $k \geq 2$, has moderate growth at cusps
- Induces a linear functional based on the **Petersson Inner Product**

$$g \to \langle g, f \rangle$$

  for $g \in S_k(\Gamma_0(N))$
- $\implies$ There is some $\widetilde{f} \in S_k(\Gamma_0(N))$ such that $\langle \cdot, f \rangle = \langle \cdot, \widetilde{f} \rangle$
- $\widetilde{f}$ is essentially the holomorphic projection of f

Holomorphic Projection

Holomorphic Projection

- ☐ If $f(z)$ is holomorphic, $\pi_{\text{hol}} f = f(z)$

Holomorphic Projection

- If $f(z)$ is holomorphic, $\pi_{\text{hol}} f = f(z)$
- $\pi_{\text{hol}}(f(z)) \in \begin{cases} M_k(\Gamma_0(N)) & k > 2 \\ M_2(\Gamma_0(N)) \oplus \mathbb{C}E_2 & k = 2 \end{cases}$

Holomorphic Projection

- ☐ If $f(z)$ is holomorphic, $\pi_{\text{hol}} f = f(z)$
- ☐ $\pi_{\text{hol}}(f(z)) \in \begin{cases} M_k(\Gamma_0(N)) & k > 2 \\ M_2(\Gamma_0(N)) \oplus \mathbb{C}E_2 & k = 2 \end{cases}$
- ☐ $\langle g, f \rangle = \langle g, \pi_{\text{hol}} f \rangle$ for $g \in S_k(\Gamma_0(N))$

Holomorphic Projection

- Consider a weight 2 newform $f_E \in S_2(\Gamma_0(N))$ associated to an elliptic curve E

Holomorphic Projection

- ☐ Consider a weight 2 newform $f_E \in S_2(\Gamma_0(N))$ associated to an elliptic curve E and the Weierstrass mock modular form $\widehat{\mathfrak{Z}}_E^+$

Holomorphic Projection

- Consider a weight 2 newform $f_E \in S_2(\Gamma_0(N))$ associated to an elliptic curve E and the Weierstrass mock modular form $\widehat{\mathfrak{Z}}_E^+$

Proposition

The holomorphic projection is given by

$$\pi_{hol}(\widehat{\mathfrak{Z}}_E^+ \cdot f_E) = \widehat{\mathfrak{Z}}_E^+ f_E - \sum_{h=1}^{\infty} q^h \underbrace{\sum_{n=1}^{\infty} a_E(n+h)\overline{a_E(n)}\left(\frac{1}{(n+h)} - \frac{1}{n}\right)}_{D_{f_E}(h;1)}$$

Holomorphic Projection

- Consider a weight 2 newform $f_E \in S_2(\Gamma_0(N))$ associated to an elliptic curve E and the Weierstrass mock modular form $\widehat{\mathfrak{Z}}_E^+$

Proposition

The holomorphic projection is given by

$$\pi_{hol}(\widehat{\mathfrak{Z}}_E^+ \cdot f_E) = \widehat{\mathfrak{Z}}_E^+ f_E - \sum_{h=1}^{\infty} q^h \underbrace{\sum_{n=1}^{\infty} a_E(n+h)\overline{a_E(n)} \left(\frac{1}{(n+h)} - \frac{1}{n} \right)}_{D_{f_E}(h;1)}$$

and $\pi_{hol}(\widehat{\mathfrak{Z}}_E^+ \cdot f_E) \in M_2(\Gamma_0(N)) \oplus \mathbb{C}E_2$.

Examples

Example [Holomorphic Projection] (CM Curves)

N	$\widehat{\pi}_{\text{hol}}(\cdot)$
27	$\frac{1}{36}\left(1-24\sum_{n=1}^{\infty}\sigma(3n)q^{3n}\right)-\frac{1}{36}\left(1+12\sum_{n=1}^{\infty}\left(\sum_{d\mid 3n, 3\nmid d} d\right)q^{3n}\right)+\left(1+3\sum_{n=1}^{\infty}\sigma(n)q^{3n}\right)$
32	$\frac{1}{48}\left(1-24\sum_{n=1}^{\infty}\sigma(4n)q^{4n}\right)+\frac{47}{48}\left(1+24\sum_{n=1}^{\infty}\left(\sum_{d\mid 4n, 2\nmid d} d\right)q^{4n}\right)$ $-\frac{7}{2}\left(1+8\sum_{n=1}^{\infty}\left(\sum_{d\mid 4n, 4\nmid d} d\right)q^{4n}\right)-\frac{15}{2}\left(1+\frac{24}{7}\sum_{n=1}^{\infty}\sigma(n)q^{4n}\right)$
36	$\frac{1}{72}\left(1-24\sum_{n=1}^{\infty}\sigma(6n)q^{6n}\right)-\frac{1}{36}\left(1+24\sum_{n=1}^{\infty}\left(\sum_{d\mid 6n, 2\nmid d} d\right)q^{6n}\right)$ $-\frac{1}{6}\left(1+12\sum_{n=1}^{\infty}\left(\sum_{d\mid 6n, 3\nmid d} d\right)q^{6n}\right)+\frac{85}{72}\left(1+\frac{24}{5}\sum_{n=1}^{\infty}\sigma(n)q^{6n}\right)$
49	$\frac{1}{56}\left(1-24\sum_{n=1}^{\infty}\sigma(7n)q^{7n}\right)+\frac{55}{56}\left(1+4\sum_{n=1}^{\infty}\left(\sum_{d\mid 7n, 7\nmid d} d\right)q^{7n}\right)$

Example (N = 11)

- Modular curve $X_0(11)$, $\dim(X_0(11)) = 1$:

$$E : y^2 + y = x^3 - x^2 - 10x - 20.$$

Example (N = 11)

- Modular curve $X_0(11)$, $\dim(X_0(11)) = 1$:

$$E : y^2 + y = x^3 - x^2 - 10x - 20.$$

- $S(\Lambda_E) = 0.38124\ldots$ gives

$$\widehat{\mathfrak{Z}}_E^+(z) = q^{-1} + 1 + 0.9520\ldots q + 1.547\ldots q^2 + 0.3493\ldots q^3 + 1.976\ldots q^4 - 2.609\ldots q^5 + O(q^6).$$

Example (N = 11)

- Modular curve $X_0(11)$, $\dim(X_0(11)) = 1$:

$$E : y^2 + y = x^3 - x^2 - 10x - 20.$$

- $S(\Lambda_E) = 0.38124\ldots$ gives

$$\widehat{\mathfrak{Z}}^+_E(z) = q^{-1} + 1 + 0.9520\ldots q + 1.547\ldots q^2 + 0.3493\ldots q^3 + 1.976\ldots q^4 - 2.609\ldots q^5 + O(q^6).$$

- 10^5 coefficients of f_E to compute $\mathbb{L}_{f_E}(z) = \sum_{n=1}^{\infty} D_{f_E}(h;1)q^h$

$$\mathbb{L}_{f_E}(z) = 0.7063\ldots q + 1.562\ldots q^2 + 0.0944\ldots q^3 + 1.237\ldots q^4 - 2.026\ldots q^5 + O(q^6)$$

Example (N = 11)

- $\dim(M_2(\Gamma_0(11)) \oplus \mathbb{C}E_2) = 2$ with basis $F^0_{11,2}$, $F^\infty_{11,2}$:

$$F^\infty_{11,2} = 1 + \frac{1}{5}q + \frac{3}{5}q^2 + \frac{4}{5}q^3 + \frac{7}{5}q^4 + \frac{6}{5}q^5 + \frac{12}{5}q^6 + O(q^7)$$

Example (N = 11)

- $\dim(M_2(\Gamma_0(11)) \oplus \mathbb{C}E_2) = 2$ with basis $F^0_{11,2}$, $F^\infty_{11,2}$:

$$F^\infty_{11,2} = 1 + \frac{1}{5}q + \frac{3}{5}q^2 + \frac{4}{5}q^3 + \frac{7}{5}q^4 + \frac{6}{5}q^5 + \frac{12}{5}q^6 + O(q^7)$$

- Apply Theorem 1 with $\alpha = .0016 \approx 0, \beta_1 = 1, \beta_2 = 0$:

$$\begin{aligned}\frac{\mathrm{vol}(\Lambda_E)}{\pi} &\left((f_E \cdot \widehat{\mathfrak{Z}}^+_E) - \alpha f_E - F^\infty_{11,2}\right) \\ &= 0.706\ldots q + 1.562\ldots q^2 + 0.0930\ldots q^3 \\ &+ 1.234\ldots q^4 - 2.024\ldots q^5 + O(q^6) \\ &= \mathbb{L}_{f_E}(z)\end{aligned}$$

Example (N = 27)

- Strong Weil curve with $N_E = 27$: $E_{27} = y^2 + y = x^3 - 7$.

Example (N = 27)

- Strong Weil curve with $N_E = 27$: $E_{27} = y^2 + y = x^3 - 7$.
- Weight 2 newform:

$$f_{E_{27}} = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + O(q^{20}).$$

Example (N = 27)

- Strong Weil curve with $N_E = 27$: $E_{27} = y^2 + y = x^3 - 7$.
- Weight 2 newform:

$$f_{E_{27}} = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + O(q^{20}).$$

- Mock modular form:

$$\widehat{\mathfrak{Z}}^+_{E_{27}}(z) = q^{-1} + \frac{1}{2}q^2 + \frac{1}{5}q^5 + \frac{3}{4}q^8 - \frac{6}{11}q^{11} - \frac{1}{2}q^{14} + O(q^{17}).$$

Example (N = 27)

- Strong Weil curve with $N_E = 27$: $E_{27} = y^2 + y = x^3 - 7$.
- Weight 2 newform:

$$f_{E_{27}} = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + O(q^{20}).$$

- Mock modular form:

$$\widehat{\mathfrak{Z}}^+_{E_{27}}(z) = q^{-1} + \frac{1}{2}q^2 + \frac{1}{5}q^5 + \frac{3}{4}q^8 - \frac{6}{11}q^{11} - \frac{1}{2}q^{14} + O(q^{17}).$$

- Holomorphic projection:

$$\widehat{\pi}_{\mathrm{hol}}(f_E \cdot \widehat{\mathfrak{Z}}^+_{E_{27}})(z) = 1 + 3q^9 + 9q^{18} - 12q^{27} \dots.$$

- Normalized $F^{\infty}_{27,2}$:

$$F^{\infty}_{27,2} = 1 + 3q^9 + 9q^{18} - 12q^{27} \dots$$

Conclusion

Summary

- ☐ For nice elliptic curves E, give an explicit characterization of shifted convolution L-series values $D_{f_E}(h;1)$ using elliptic curve invariants.

Summary

- ☐ For nice elliptic curves E, give an explicit characterization of shifted convolution L-series values $D_{f_E}(h; 1)$ using elliptic curve invariants.
- ☐ Enable fast computation of $\mathbb{L}_{f_E}(z)$ coefficients.

Summary

- ☐ For nice elliptic curves E, give an explicit characterization of shifted convolution L-series values $D_{f_E}(h;1)$ using elliptic curve invariants.
- ☐ Enable fast computation of $\mathbb{L}_{f_E}(z)$ coefficients.
- ☐ Understand properties of Weierstrass mock modular form $\widehat{\mathfrak{Z}}_E^+(z)$ in terms of f_E, invariants of $E/\mathbb{Q}$.

Summary

- ☐ For nice elliptic curves E, give an explicit characterization of shifted convolution L-series values $D_{f_E}(h;1)$ using elliptic curve invariants.
- ☐ Enable fast computation of $\mathbb{L}_{f_E}(z)$ coefficients.
- ☐ Understand properties of Weierstrass mock modular form $\widehat{\mathfrak{Z}}_E^+(z)$ in terms of f_E, invariants of $E/\mathbb{Q}$.
- ☐ Understand support, arithmetic properties of Fourier coefficients of $f_E, \widehat{\mathfrak{Z}}_E^+$ for CM curves E

Summary

- ☐ For nice elliptic curves E, give an explicit characterization of shifted convolution L-series values $D_{f_E}(h; 1)$ using elliptic curve invariants.
- ☐ Enable fast computation of $\mathbb{L}_{f_E}(z)$ coefficients.
- ☐ Understand properties of Weierstrass mock modular form $\widehat{\mathfrak{Z}}_E^+(z)$ in terms of f_E, invariants of $E/\mathbb{Q}$.
- ☐ Understand support, arithmetic properties of Fourier coefficients of $f_E, \widehat{\mathfrak{Z}}_E^+$ for CM curves E
- ☐ Future work: generalize to other elliptic curves E, understand other L-series values + interesting variants.

Acknowledgements

Thanks to the NSF and the Emory REU, in particular Ken Ono and Michael Mertens for all of their support with our projects. Thanks to the anonymous Archiv der Mathematik referee for valuable feedback on this work.

Questions?