West Coast Number Theory, Pacific Grove, CA, December 2017

Perfect squares as concatenation of consecutive integers

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Concatenations as perfect squares I

- In 1998, Sastry noticed that $183184 = 428^2$ and asked if there are other examples of positive integers a such that concatenating a with $a + 1$ (from left to right) in base 10 results in a perfect square.
- Let n be the number of digits of $a + 1$; the question reduces to finding other instances when

$$10^n a + (a + 1) = x^2 \tag{1}$$

with positive integers $a,\ x$, $10^{n-1} \leq a < 10^n - 1$. Equivalently,

$$a(10^n + 1) = x^2 - 1 = (x - 1)(x + 1).$$

Concatenations as perfect squares II

- The arithmetic of $10^n + 1$ plays a role: if $10^n + 1$ is a prime, then the above equation implies that $10^n + 1$ divides one of $x - 1$ or $x + 1$. Thus, $x + 1 \geq 10^n + 1$, so $x^2 \geq 10^{2n}$ is a number with at least $2n + 1$ digits; a contradiction with the fact that it should have exactly $2n$ digits.
- $10^n + 1$ $(n > 1)$ is never a perfect power (shown easily in an elementary way or invoking known facts about Catalan's equation); it follows that if $10^n + 1$ is not prime, then it has at least two distinct prime factors.
- Now, we write $10^n + 1 = A_1 A_2$ and $a = a_1 a_2$ and try to solve

$$x + 1 = A_1 a_1 \quad \text{and} \quad x - 1 = A_2 a_2,$$

implying

$$A_1 a_1 - A_2 a_2 = 2. \tag{2}$$

Concatenations as perfect squares III

- Since A_1, A_2 are odd (as divisors of $10^n + 1$), we deduce from (2) that they must be coprime and (from the previous argument) none of A_1, A_2 can be 1. Given A_1, A_2, equation (2) has infinitely many solutions (a_1, a_2), coming from the minimal one, let's call it $(a_{1,0}, a_{2,0})$, via

$$a_1 = a_{1,0} + A_2 m \quad \text{and} \quad a_2 = a_{2,0} + A_1 m, m \in \mathbb{Z}.$$

- Since $a_1 a_2 = a < 10^n - 1 < A_1 A_2$, then $(a_1, a_2) = (a_{1,0}, a_{2,0})$. If $a_0 = a_{1,0} a_{2,0}$ has n digits, we found a convenient solution to our problem.

Concatenations as perfect squares IV

- Sometimes, a_0 is "too short". For example, taking $m = 3,\ A_1 = 11,\ A_2 = 91$, the minimal solution of the equation

$$11a_1 - 91a_2 = 2$$

  is $(a_{1,0}, a_{2,0}) = (25, 3)$ for which $a_0 = 75$ has only two digits.
- If we pretend that it has three digits, namely that it is 075, then indeed concatenating a_0 with $a_0 + 1$ results in the perfect square

$$075076 = 274^2.$$

Concatenations as perfect squares V

- Note that if (a_1, a_2) is a solution of (2), then $(a_1', a_2') = (A_2 - a_1, A_1 - a_2)$ is a solution of

$$A_1 a_1' - A_2 a_2' = -2,$$

  which is the same equation as (2) with the pair (A_1, A_2) replaced by the pair (A_2, A_1).
- One can show that given A_1, A_2, not both a_0 and a_0' can be short.
- For the example with $m = 3$, $A_1 = 11$, $A_2 = 91$, we have $(a_{1,0}', a_{2,0}') = (66, 8)$, so $a_0' = 66 \times 8 = 528$ has three digits and

$$528529 = 727^2.$$

Concatenations as perfect squares VI

- As a byproduct, one finds that if one denotes by $N_+(n) := \#\{a : a||(a+1) = \Box, \textit{a has n digits}\}$, then $N_+(n) \neq 0$ if and only if $10^n + 1$ has at least two distinct prime factors.
- Let $\omega(n)$ be the number of distinct prime factors of $10^n + 1$; one can show that

$$2^{\omega(n)-1} - 1 \leq N_+(n) \leq 2(2^{\omega(n)-1} - 1).$$

- We notice that

$$66 \times 8, \qquad 6666 \times 68, \qquad 666666 \times 668, \qquad \ldots$$

all work as integers a, with $a||(a+1) = \Box$.

Concatenations as perfect squares VII

- We conjecture and then show that for all m, the number

$$a = \underbrace{66\ldots6}_{2m\text{ times}} \times \underbrace{66\ldots6}_{m-1\text{ times}}8 \tag{3}$$

is a valid example with $3m$ digits: for such values of a, then $a||(a+1)$ is a polynomial of deg 6 in 10^m, the square of a polynomial of degree 3 in 10^m.

Concatenations as perfect squares VIII

- I. Shparlinski noticed that the above problem was easy because $x^2 - 1$ factors as $(x-1)(x+1)$, so he asked, what about if we concatenate a with $a+1$ in the reverse order and ask for that to be a square.
- The analog equation is then

$$10^n a + (a-1) = x^2 \Longleftrightarrow a(10^n + 1) = x^2 + 1. \qquad (4)$$

- Then, n is even, since if odd, then $11 | x^2 + 1$, so $x^2 \equiv -1 \pmod{11}$, and this is impossible. This argument also shows that all prime factors of both a and $10^n + 1$ are congruent to 1 modulo 4.

Concatenations as perfect squares IX

- Factor $x^2 + 1 = (x + i)(x - i)$, so $x + i \mid a(10^n + 1)$ in $\mathbb{Z}[i]$. Then $\exists\ a_1, a_2, A_1, A_2 \in \mathbb{Z}$ such that

$$x + i = (a_2 + a_1 i)(A_1 - A_2 i), \tag{5}$$

with $a_2 + a_1 i = \gcd(a, x + i)$ and $A_1 - A_2 i = \gcd(x + i, 10^n + 1)$ in $\mathbb{Z}[i]$.

- We may assume that A_1 and A_2 are positive, so

$$x^2+1 = (a_1^2+a_2^2)(A_1^2+A_2^2) \text{ with } a_1^2+a_2^2 = a, A_1^2+A_2^2 = 10^n+1.$$

- In (5) we identify the imaginary part from the two sides of the equation getting

$$a_1 A_1 - a_2 A_2 = 1. \tag{6}$$

Concatenations as perfect squares X

- Let $(a_{1,0}, a_{2,0})$ be its minimal solution. Then

$$(a_1, a_2) = (a_{1,0} + A_2 m, a_{2,0} + A_1 m), \quad \text{for some} \quad m \geq 0.$$

Then, if $m \geq 1$,

$$a = a_1^2 + a_2^2 > (A_1^2 + A_2^2)m^2 \geq A_1^2 + A_2^2 > 10^n$$

is “too long". Hence, the only chance is that $(a_1, a_2) = (a_{1,0}, a_{2,0})$.

- One can take $a_0 = a_{1,0}^2 + a_{2,0}^2$. If a_0 is “too short", that is, $a_0 < 10^{n-1}$. But then, the pair $(a_1', a_2') = (A_2 - a_1, A_2 - a_2)$ satisfies

$$a_1' A_1 - a_2' A_2 = -1,$$

and we showed one of these situations will give the right number of digits.

Concatenations as perfect squares XI

- Recall that the number of representations as a sum of two squares of $10^n + 1$ equals $2^{\omega(n)}$, and letting $N_-(n)$ be the number of positive integers a with n digits satisfying Shparlinski's requirement, we can show:

Theorem (Luca-S. 2017)

Let $n \geq 1$ be a positive integer. Then $N_-(n) = 0$ unless n is even. Furthermore, the inequality

$$2^{\omega(n)-1} \leq N_-(n) \leq 2(2^{\omega(n)-1} - 1) + 1$$

holds for all even n.

Concatenations as perfect squares XII

- How about finding parametric families of solutions?
- Taking $n = 6k$ and giving k values $1, 2, 3$, one gets the examples $146^2 + 719^2,\ 13466^2 + 673199^2,\ 1334666^2 + 667331999^2, \cdots$ inferring that perhaps (a_1, a_2), where

$$a_1 = 1\underbrace{33\ldots3}_{k-1\text{ times}}4\underbrace{66\ldots6}_{k\text{ times}}$$
$$a_2 = \underbrace{66\ldots6}_{k-1\text{ times}}7\underbrace{33\ldots3}_{k-1\text{ times}}1\underbrace{99\ldots9}_{k\text{ times}}$$

has the property that $a = a_1^2 + a_2^2$ is a valid solution (with n digits) to Shparlinski's question.

Concatenations as perfect squares XIII

- One checks easily that

$$
\begin{aligned}
a_1 &= \frac{4 \cdot 10^{2k} + 4 \cdot 10^{k} - 2}{3} \\
a_2 &= \frac{2 \cdot 10^{3k} + 2 \cdot 10^{2k} - 4 \cdot 10^{k} - 3}{3}
\end{aligned}
$$

and indeed

$$
a = a_1^2 + a_2^2 = \left(\frac{2 \cdot 10^{6k} + 2 \cdot 10^{5k} + 10^{3k} + 2 \cdot 10^{k} + 2}{3} \right)^2
$$

is a perfect square.

Concatenations as perfect squares XIV

- We also found a parametric family for $n = 10k$, and even a "short parametric family" for such n, where by short we mean that a has $8k$ digits, instead of $10k$, so it has to be "beefed up" by $2k$ zeros to the left in order to create an example. The "short parametric family" is given by

$$a_1 = 7\underbrace{99\ldots9}_{2(k-1)\text{ times}}84\underbrace{00\ldots0}_{k-1\text{ times}} = \frac{4\cdot10^{3k}-8\cdot10^{k}}{5}$$

$$a_2 = 3\underbrace{99\ldots9}_{2(k-1)\text{ times}}88\underbrace{00\ldots0}_{k-1\text{ times}}1 = \frac{2\cdot10^{4k}-6\cdot10^{2k}+5}{5},$$

which, of course, can be changed into the "right" one by the previously mentioned trick. We leave the details to the interested audience.

Concatenations as perfect squares XV

We conclude this discussion with the following open problem for the audience.

Problem (Luca-S. 2017)

For what integer values d, are there infinitely many positive integers a such that a and $a + d$ have the same number of digits and concatenating a with $a + d$ (from left to right) one gets a perfect square?

- In this paper, we treated the cases $d = \pm 1$. The case $d = 0$ is related to $10^n + 1$ not being square-free.

Concatenations as perfect squares XVI

- In this case, the analog equation (1) is

$$a(10^n + 1) = x^2,$$

  and if $10^n + 1$ is squarefree, then $10^n + 1 \mid x$, which implies $a(10^n + 1) \geq (10^n + 1)^2$, so $a > 10^n$, a contradiction.
- For example, for $n = 11$, $10^{11} + 1$ is a multiple of 11^2, and taking

$$a = \left(\frac{10^{11} + 1}{11^2}\right) y^2$$

Concatenations as perfect squares XVII

for some integer y such that a has exactly 11 digits, one gets the examples

$$
\begin{aligned}
13223140496\,13223140496 &= 36363636364^2, & y = 4;\\
20661157025\,20661157025 &= 45454545455^2, & y = 5;\\
29752066116\,29752066116 &= 54545454546^2, & y = 6;\\
40495867769\,40495867769 &= 63636363637^2, & y = 7;\\
52892561984\,52892561984 &= 72727272728^2, & y = 8;\\
66942148761\,66942148761 &= 81818181819^2, & y = 9;\\
82644628100\,82644628100 &= 90909090910^2, & y = 10.
\end{aligned}
$$

Concatenations as perfect squares XVIII

- More generally, if $m^2 > 1$ is any square factor of $10^n + 1$, then taking

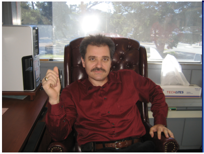
$$a = \left(\frac{10^n + 1}{m^2}\right) y^2$$

  with an integer y in the interval $[m/\sqrt{10}, m-1]$, gives a valid answer to our problem for $d = 0$.

- How about for other values of d? One can quickly check that for all d with $|d| \leq 10$, one can find examples of perfect squares by concatenating a with $a + d$ from left to right, except for $d = -3, 7$ (with a little modular arithmetic work, one can give an argument why those values of d will never generate perfect squares).

Concatenations as perfect squares XIX

- Certainly, one can ask similar questions of concatenating a sequence of consecutive integers (all with the same number of digits) in some order, giving rise to a perfect square, which questions we invite the audience to investigate.



Theorem (Pante Stanica)

Thank you for your attention!

Proof.

None required! □