2-adic valuations of generalized Fibonacci sequences

Paul Thomas Young

College of Charleston

December 17, 2017

Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, . . .

Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, . . .

Every third Fibonacci number is even.

Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, . . .

Every third Fibonacci number is even. In fact we have the following:

Theorem

The 2-adic valuation of the n-th Fibonacci number is given by

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod 3, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, . . .

Every third Fibonacci number is even. In fact we have the following:

Theorem

The 2-adic valuation of the n-th Fibonacci number is given by

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{3}, \\ 1, & \text{if } n \equiv 3 \pmod{6}, \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{6}. \end{cases}$$

There are similar results for other primes. For example, $\nu_5(F_n) = \nu_5(n)$.

Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, . . .

Every third Fibonacci number is even. In fact we have the following:

Theorem

The 2-adic valuation of the n-th Fibonacci number is given by

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod 3, \\ 1, & \text{if } n \equiv 3 \pmod 6, \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

There are similar results for other primes. For example, $\nu_5(F_n) = \nu_5(n)$.

We consider the 2-adic valuation of the generalized Fibonacci sequence (T_n) of order k, defined by the recurrence $T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}$ for $n \geq k$, with initial conditions $T_0 = 0$ and $T_i = 1$ for $1 \leq i < k$. We are motivated by two recent conjectures of Lengyel and Marques.

$k = 3$: 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, . . .

$k = 4$: 0, 1, 1, 1, 3, 6, 11, 21, 41, 79, 152, 293, 565, 1089, 2099, 4046, 7799, . . .

$k = 5$: 0, 1, 1, 1, 1, 4, 8, 15, 29, 57, 113, 222, 436, 857, 1685, 3313, 6513, . . .

Tribonacci and Tetranacci sequences ($k = 3, 4$)

Theorem (Marques and Lengyel, 2014)

For order $k = 3$, the 2-adic valuation of the n-th Tribonacci number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 4, \\ 1, & \text{if } n \equiv 3, 11 \pmod{16}, \\ 2, & \text{if } n \equiv 4, 8 \pmod{16}, \\ 3, & \text{if } n \equiv 7 \pmod{16}, \\ \nu_2(n) - 1, & \text{if } n \equiv 0 \pmod{16}, \\ \nu_2(n+4) - 1, & \text{if } n \equiv 12 \pmod{16}, \\ \nu_2((n+1)(n+17)) - 3, & \text{if } n \equiv 15 \pmod{16}. \end{cases}$$

Tribonacci and Tetranacci sequences ($k = 3, 4$)

Theorem (Marques and Lengyel, 2014)

For order $k = 3$, the 2-adic valuation of the n-th Tribonacci number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 4, \\ 1, & \text{if } n \equiv 3, 11 \pmod{16}, \\ 2, & \text{if } n \equiv 4, 8 \pmod{16}, \\ 3, & \text{if } n \equiv 7 \pmod{16}, \\ \nu_2(n) - 1, & \text{if } n \equiv 0 \pmod{16}, \\ \nu_2(n+4) - 1, & \text{if } n \equiv 12 \pmod{16}, \\ \nu_2((n+1)(n+17)) - 3, & \text{if } n \equiv 15 \pmod{16}. \end{cases}$$

The method of proof was elementary, by means of several congruence results. They used this result to solve the Diophantine equation $T_n = m!$.

Tribonacci and Tetranacci sequences ($k = 3, 4$)

Theorem (Marques and Lengyel, 2014)

For order $k = 3$, the 2-adic valuation of the n-th Tribonacci number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 4, \\ 1, & \text{if } n \equiv 3, 11 \pmod{16}, \\ 2, & \text{if } n \equiv 4, 8 \pmod{16}, \\ 3, & \text{if } n \equiv 7 \pmod{16}, \\ \nu_2(n) - 1, & \text{if } n \equiv 0 \pmod{16}, \\ \nu_2(n+4) - 1, & \text{if } n \equiv 12 \pmod{16}, \\ \nu_2((n+1)(n+17)) - 3, & \text{if } n \equiv 15 \pmod{16}. \end{cases}$$

The method of proof was elementary, by means of several congruence results. They used this result to solve the Diophantine equation $T_n = m!$.

Theorem (Lengyel and Marques, 2017)

For order $k = 4$, the 2-adic valuation of the n-th Tetranacci number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod 5, \\ 1, & \text{if } n \equiv 5 \pmod{10}, \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{10}. \end{cases}$$

Order $k > 4$

In their 2017 paper Lengyel and Marques conjectured that for order $k \geq 3$, $\nu_2(T_{s(k+1)2^r}) = r + \nu_2(k-2) + 1$ for integers $r \geq 1$ and s odd.

Order $k > 4$

In their 2017 paper Lengyel and Marques conjectured that for order $k \geq 3$, $\nu_2(T_{s(k+1)2^r}) = r + \nu_2(k-2) + 1$ for integers $r \geq 1$ and s odd.

This was immediately proved by Sobolewski in the case of even k:

Theorem (Sobolewski, 2017)

For even order $k \geq 4$, the 2-adic valuation of the n-th generalized Fibonacci number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{k+1}, \\ 1, & \text{if } n \equiv k+1 \pmod{2k+2}, \\ \nu_2(n) + \nu_2(k-2) + 1, & \text{if } n \equiv 0 \pmod{2k+2}. \end{cases}$$

Order $k > 4$

In their 2017 paper Lengyel and Marques conjectured that for order $k \geq 3$, $\nu_2(T_{s(k+1)2^r}) = r + \nu_2(k-2) + 1$ for integers $r \geq 1$ and s odd.

This was immediately proved by Sobolewski in the case of even k:

Theorem (Sobolewski, 2017)

For even order $k \geq 4$, the 2-adic valuation of the n-th generalized Fibonacci number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{k+1}, \\ 1, & \text{if } n \equiv k+1 \pmod{2k+2}, \\ \nu_2(n) + \nu_2(k-2) + 1, & \text{if } n \equiv 0 \pmod{2k+2}. \end{cases}$$

The method of proof was based on an intricate system of congruences, which are critically dependent on the assumption that k is even.

Order $k > 4$

In their 2017 paper Lengyel and Marques conjectured that for order $k \geq 3$, $\nu_2(T_{s(k+1)2^r}) = r + \nu_2(k-2) + 1$ for integers $r \geq 1$ and s odd.

This was immediately proved by Sobolewski in the case of even k:

Theorem (Sobolewski, 2017)

For even order $k \geq 4$, the 2-adic valuation of the n-th generalized Fibonacci number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{k+1}, \\ 1, & \text{if } n \equiv k+1 \pmod{2k+2}, \\ \nu_2(n) + \nu_2(k-2) + 1, & \text{if } n \equiv 0 \pmod{2k+2}. \end{cases}$$

The method of proof was based on an intricate system of congruences, which are critically dependent on the assumption that k is even.

Sobolewski's work was motivated by the problem of showing that the Diophantine equation

$$m! = T_{n_1} \cdots T_{n_d}$$

has only finitely many solutions.

Order $k > 4$

In their 2017 paper Lengyel and Marques conjectured that for order $k \geq 3$, $\nu_2(T_{s(k+1)2^r}) = r + \nu_2(k-2) + 1$ for integers $r \geq 1$ and s odd.

This was immediately proved by Sobolewski in the case of even k:

Theorem (Sobolewski, 2017)

For even order $k \geq 4$, the 2-adic valuation of the n-th generalized Fibonacci number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod{k+1}, \\ 1, & \text{if } n \equiv k+1 \pmod{2k+2}, \\ \nu_2(n) + \nu_2(k-2) + 1, & \text{if } n \equiv 0 \pmod{2k+2}. \end{cases}$$

The method of proof was based on an intricate system of congruences, which are critically dependent on the assumption that k is even.

Sobolewski's work was motivated by the problem of showing that the Diophantine equation

$$m! = T_{n_1} \cdots T_{n_d}$$

has only finitely many solutions.

His theorem also shows that, for even order k, the sequence $(\nu_2(T_n))$ is a 2-regular sequence.

Order $k = 5$

Theorem (Lengyel and Marques, 2017)

For order $k = 5$, the 2-adic valuation of the n-th generalized Fibonacci number number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, 5 \pmod 6, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ \nu_2(n), & \text{if } n \equiv 0 \pmod{12}, \\ \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n-6) \neq 3. \end{cases}$$

Order $k = 5$

Theorem (Lengyel and Marques, 2017)

For order $k = 5$, the 2-adic valuation of the n-th generalized Fibonacci number number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, 5 \pmod 6, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ \nu_2(n), & \text{if } n \equiv 0 \pmod{12}, \\ \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n-6) \neq 3. \end{cases}$$

What is the deal with the case $n \equiv 6 \pmod{12}$?

Order $k = 5$

Theorem (Lengyel and Marques, 2017)

For order $k = 5$, the 2-adic valuation of the n-th generalized Fibonacci number number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, 5 \pmod 6, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ \nu_2(n), & \text{if } n \equiv 0 \pmod{12}, \\ \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n-6) \neq 3. \end{cases}$$

What is the deal with the case $n \equiv 6 \pmod{12}$? They made a conjecture:

Conjecture (L,M 2017)

For order $k = 5$, the 2-adic valuation of T_n for $n \equiv 6 \pmod{12}$ satisfies

$$\nu_2(T_n) = \begin{cases} \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) < 8, \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) \geq 8. \end{cases}$$

Order $k = 5$

Theorem (Lengyel and Marques, 2017)

For order $k = 5$, the 2-adic valuation of the n-th generalized Fibonacci number number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0,5 \pmod 6, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ \nu_2(n), & \text{if } n \equiv 0 \pmod{12}, \\ \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n-6) \neq 3. \end{cases}$$

What is the deal with the case $n \equiv 6 \pmod{12}$? They made a conjecture:

Conjecture (L,M 2017)

For order $k = 5$, the 2-adic valuation of T_n for $n \equiv 6 \pmod{12}$ satisfies

$$\nu_2(T_n) = \begin{cases} \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \end{cases}$$

Theorem (Lengyel and Marques, 2017)

For order $k = 5$, the 2-adic valuation of the n-th generalized Fibonacci number number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, 5 \pmod 6, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ \nu_2(n), & \text{if } n \equiv 0 \pmod{12}, \\ \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n-6) \neq 3. \end{cases}$$

What is the deal with the case $n \equiv 6 \pmod{12}$? They made a conjecture:

Conjecture (L,M 2017)

For order $k = 5$, the 2-adic valuation of T_n for $n \equiv 6 \pmod{12}$ satisfies

$$\nu_2(T_n) = \begin{cases} \\ \nu_2(n + 43266), & \text{if } n \equiv 6 \pmod{12} \end{cases}$$

Really?

Order $k = 5$

Theorem (Lengyel and Marques, 2017)

For order $k = 5$, the 2-adic valuation of the n-th generalized Fibonacci number number T_n is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, 5 \pmod 6, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ \nu_2(n), & \text{if } n \equiv 0 \pmod{12}, \\ \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n-6) \neq 3. \end{cases}$$

What is the deal with the case $n \equiv 6 \pmod{12}$? They made a conjecture:

Conjecture (L,M 2017)

For order $k = 5$, the 2-adic valuation of T_n for $n \equiv 6 \pmod{12}$ satisfies

$$\nu_2(T_n) = \begin{cases} \\ \nu_2(n + 43266), & \text{if } n \equiv 6 \pmod{12} \end{cases}$$

Really?

Despite my disbelief, this conjecture is correct for positive integers n less than three million.

Interpolating subsequences of (T_n) 2-adically

Theorem (PTY, 2017)

Write $k+1 = 2^e l$ with l odd. Then for each $j \in \mathbb{Z}$ there exists a continuous function $f_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ such that $f_j(n) = T_{ln+j}$ for all $n \in \mathbb{Z}$. Furthermore, for each $j \in \mathbb{Z}$ there exists a function g_j which is analytic on $D = \{x \in \mathbb{C}_2 : \nu_2(x) > -1\}$ such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$.

Theorem (PTY, 2017)

Write $k+1 = 2^e l$ with l odd. Then for each $j \in \mathbb{Z}$ there exists a continuous function $f_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ such that $f_j(n) = T_{ln+j}$ for all $n \in \mathbb{Z}$. Furthermore, for each $j \in \mathbb{Z}$ there exists a function g_j which is analytic on $D = \{x \in \mathbb{C}_2 : \nu_2(x) > -1\}$ such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$.

The proof of this theorem is based on the Binet formula

$$T_n = c_1\alpha_1^n + \cdots + c_k\alpha_k^n$$

where the α_i are the roots of the charactistic polynomial.

Theorem (PTY, 2017)

Write $k+1 = 2^e l$ with l odd. Then for each $j \in \mathbb{Z}$ there exists a continuous function $f_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ such that $f_j(n) = T_{ln+j}$ for all $n \in \mathbb{Z}$. Furthermore, for each $j \in \mathbb{Z}$ there exists a function g_j which is analytic on $D = \{x \in \mathbb{C}_2 : \nu_2(x) > -1\}$ such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$.

The proof of this theorem is based on the Binet formula

$$T_n = c_1\alpha_1^n + \cdots + c_k\alpha_k^n$$

where the α_i are the roots of the charactistic polynomial.

For $z \in \mathbb{C}_2$, the exponential function

$$(1+z)^x := \sum_{m=0}^{\infty} \binom{x}{m} z^m$$

is a continuous function of $x \in \mathbb{Z}_2$ when $\nu_2(z) > 0$,

Interpolating subsequences of (T_n) 2-adically

Theorem (PTY, 2017)

Write $k+1 = 2^e l$ with l odd. Then for each $j \in \mathbb{Z}$ there exists a continuous function $f_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ such that $f_j(n) = T_{ln+j}$ for all $n \in \mathbb{Z}$. Furthermore, for each $j \in \mathbb{Z}$ there exists a function g_j which is analytic on $D = \{x \in \mathbb{C}_2 : \nu_2(x) > -1\}$ such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$.

The proof of this theorem is based on the Binet formula

$$T_n = c_1\alpha_1^n + \cdots + c_k\alpha_k^n$$

where the α_i are the roots of the charactistic polynomial.

For $z \in \mathbb{C}_2$, the exponential function

$$(1+z)^x := \sum_{m=0}^{\infty} \binom{x}{m} z^m = \exp_2(x \log_2(1+z))$$

is a continuous function of $x \in \mathbb{Z}_2$ when $\nu_2(z) > 0$, and an analytic function of $x \in \mathbb{Z}_2$ when $\nu_2(z) > 1$. So we have to locate the zeros of the charactistic polynomial in $\mathbb{C}_2$.

Theorem (PTY, 2017)

Write $k+1 = 2^e l$ with l odd. Then for each $j \in \mathbb{Z}$ there exists a continuous function $f_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ such that $f_j(n) = T_{ln+j}$ for all $n \in \mathbb{Z}$. Furthermore, for each $j \in \mathbb{Z}$ there exists a function g_j which is analytic on $D = \{x \in \mathbb{C}_2 : \nu_2(x) > -1\}$ such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$.

The proof of this theorem is based on the Binet formula

$$T_n = c_1\alpha_1^n + \cdots + c_k\alpha_k^n$$

where the α_i are the roots of the charactistic polynomial.

For $z \in \mathbb{C}_2$, the exponential function

$$(1+z)^x := \sum_{m=0}^{\infty} \binom{x}{m} z^m = \exp_2(x \log_2(1+z))$$

is a continuous function of $x \in \mathbb{Z}_2$ when $\nu_2(z) > 0$, and an analytic function of $x \in \mathbb{Z}_2$ when $\nu_2(z) > 1$. So we have to locate the zeros of the charactistic polynomial in $\mathbb{C}_2$.

Corollary

The sequence (T_n) may be extended to a continuous function of $n \in \mathbb{Z}_2$ if and only if the order k is of the form $k = 2^e - 1$.

Locating the roots of the characteristic polynomial

The characteristic polynomial of the recurrence for (T_n) is

$$p(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = \frac{x^{k+1} - 2x^k + 1}{x - 1} = \frac{q(x)}{x - 1}.$$

Locating the roots of the characteristic polynomial

The characteristic polynomial of the recurrence for (T_n) is

$$p(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = \frac{x^{k+1} - 2x^k + 1}{x - 1} = \frac{q(x)}{x-1}.$$

We simplify the recurrence by treating it as order $k+1$. Note that

$$x^{k+1} - 2x^k + 1 \equiv x^{k+1} - 1 \equiv (x^l - 1)^{2^e} \pmod 2.$$

Locating the roots of the characteristic polynomial

The characteristic polynomial of the recurrence for (T_n) is

$$p(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = \frac{x^{k+1} - 2x^k + 1}{x-1} = \frac{q(x)}{x-1}.$$

We simplify the recurrence by treating it as order $k+1$. Note that

$$x^{k+1} - 2x^k + 1 \equiv x^{k+1} - 1 \equiv (x^l - 1)^{2^e} \pmod 2.$$

Proposition.

Write $k+1 = 2^e l$ with l odd. Corresponding to each nontrivial solution $\zeta \in \mathbb{C}_2$ to $\zeta^l = 1$ there are 2^e roots α of $p(x)$ which satisfy $\nu_2(\alpha - \zeta) = 2^{-e}$. Corresponding to the trivial solution $\zeta = 1$, there are $2^e - 1$ roots α of $p(x)$ which satisfy $\nu_2(\alpha - 1) > 0$. When $e = 1$, this root satisfies $\nu_2(\alpha - 1) = \nu_2(k-1)$; when $e > 1$ these $2^e - 1$ roots all satisfy $\nu_2(\alpha - 1) = (2^e - 1)^{-1}$. If α_i, α_j are two roots of $p(x)$ which correspond to the same ζ, then $\nu_2(\alpha_i - \alpha_j) = (2^e - 1)^{-1}$; otherwise $\nu_2(\alpha_i - \alpha_j) = 0$ for roots α_i, α_j of $p(x)$ corresponding to distinct solutions to $\zeta^l = 1$.

Locating the roots of the characteristic polynomial

The characteristic polynomial of the recurrence for (T_n) is

$$p(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = \frac{x^{k+1} - 2x^k + 1}{x - 1} = \frac{q(x)}{x - 1}.$$

We simplify the recurrence by treating it as order $k + 1$. Note that

$$x^{k+1} - 2x^k + 1 \equiv x^{k+1} - 1 \equiv (x^l - 1)^{2^e} \pmod 2.$$

Proposition.

Write $k + 1 = 2^e l$ with l odd. Corresponding to each nontrivial solution $\zeta \in \mathbb{C}_2$ to $\zeta^l = 1$ there are 2^e roots α of $p(x)$ which satisfy $\nu_2(\alpha - \zeta) = 2^{-e}$. Corresponding to the trivial solution $\zeta = 1$, there are $2^e - 1$ roots α of $p(x)$ which satisfy $\nu_2(\alpha - 1) > 0$. When $e = 1$, this root satisfies $\nu_2(\alpha - 1) = \nu_2(k - 1)$; when $e > 1$ these $2^e - 1$ roots all satisfy $\nu_2(\alpha - 1) = (2^e - 1)^{-1}$. If α_i, α_j are two roots of $p(x)$ which correspond to the same ζ, then $\nu_2(\alpha_i - \alpha_j) = (2^e - 1)^{-1}$; otherwise $\nu_2(\alpha_i - \alpha_j) = 0$ for roots α_i, α_j of $p(x)$ corresponding to distinct solutions to $\zeta^l = 1$.

This is proved using the theory of Newton polygons.

Locating the roots of the characteristic polynomial

The characteristic polynomial of the recurrence for (T_n) is

$$p(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = \frac{x^{k+1} - 2x^k + 1}{x - 1} = \frac{q(x)}{x - 1}.$$

We simplify the recurrence by treating it as order $k + 1$. Note that

$$x^{k+1} - 2x^k + 1 \equiv x^{k+1} - 1 \equiv (x^l - 1)^{2^e} \pmod 2.$$

Proposition.

Write $k + 1 = 2^e l$ with l odd. Corresponding to each nontrivial solution $\zeta \in \mathbb{C}_2$ to $\zeta^l = 1$ there are 2^e roots α of $p(x)$ which satisfy $\nu_2(\alpha - \zeta) = 2^{-e}$. Corresponding to the trivial solution $\zeta = 1$, there are $2^e - 1$ roots α of $p(x)$ which satisfy $\nu_2(\alpha - 1) > 0$. When $e = 1$, this root satisfies $\nu_2(\alpha - 1) = \nu_2(k - 1)$; when $e > 1$ these $2^e - 1$ roots all satisfy $\nu_2(\alpha - 1) = (2^e - 1)^{-1}$. If α_i, α_j are two roots of $p(x)$ which correspond to the same ζ, then $\nu_2(\alpha_i - \alpha_j) = (2^e - 1)^{-1}$; otherwise $\nu_2(\alpha_i - \alpha_j) = 0$ for roots α_i, α_j of $p(x)$ corresponding to distinct solutions to $\zeta^l = 1$.

This is proved using the theory of Newton polygons.

Since the α_i are close to *l-th roots of unity*, the functions $n \to \alpha_i^{ln}$ can be (continuously) interpolated, and the functions $n \to \alpha_i^{2(k+1)n}$ are analytic.

Locating the roots of the characteristic polynomial

The characteristic polynomial of the recurrence for (T_n) is

$$p(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = \frac{x^{k+1} - 2x^k + 1}{x-1} = \frac{q(x)}{x-1}.$$

We simplify the recurrence by treating it as order $k+1$. Note that

$$x^{k+1} - 2x^k + 1 \equiv x^{k+1} - 1 \equiv (x^l - 1)^{2^e} \pmod 2.$$

Proposition.

Write $k+1 = 2^e l$ with l odd. Corresponding to each nontrivial solution $\zeta \in \mathbb{C}_2$ to $\zeta^l = 1$ there are 2^e roots α of $p(x)$ which satisfy $\nu_2(\alpha - \zeta) = 2^{-e}$. Corresponding to the trivial solution $\zeta = 1$, there are $2^e - 1$ roots α of $p(x)$ which satisfy $\nu_2(\alpha - 1) > 0$. When $e = 1$, this root satisfies $\nu_2(\alpha - 1) = \nu_2(k-1)$; when $e > 1$ these $2^e - 1$ roots all satisfy $\nu_2(\alpha - 1) = (2^e - 1)^{-1}$. If α_i, α_j are two roots of $p(x)$ which correspond to the same ζ, then $\nu_2(\alpha_i - \alpha_j) = (2^e - 1)^{-1}$; otherwise $\nu_2(\alpha_i - \alpha_j) = 0$ for roots α_i, α_j of $p(x)$ corresponding to distinct solutions to $\zeta^l = 1$.

This is proved using the theory of Newton polygons.

Since the α_i are close to *l-th roots of unity*, the functions $n \to \alpha_i^{ln}$ can be (continuously) interpolated, and the functions $n \to \alpha_i^{2(k+1)n}$ are analytic.

We also need to know how close the roots α_i are to *each other*, in order to 2-adically bound the constants c_i in the Binet form $T_n = \sum_{i=1}^{k} c_i \alpha_i^n$.

Analysis of the interpolating functions

For each $j \in \mathbb{Z}$, there is an analytic function g_j such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$. Write $g_j(x) = \sum_m a_m x^m$. Then *a priori*

$$\nu_2(a_m) \geq \begin{cases} m + S_2(m) - 1, & k \text{ odd,} \\ m + S_2(m), & k \text{ even.} \end{cases}$$

Analysis of the interpolating functions

For each $j \in \mathbb{Z}$, there is an analytic function g_j such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$. Write $g_j(x) = \sum_m a_m x^m$. Then *a priori*

$$\nu_2(a_m) \geq \begin{cases} m + S_2(m) - 1, & k \text{ odd}, \\ m + S_2(m), & k \text{ even}. \end{cases}$$

We will primarily focus on the case where k is odd, since the even k case is similar, but easier. It is immediate that $T_j = g_j(0) = a_0$. In general one may approximate the coefficients a_m by computing $g_j(n)$ for several integers n and solving a system of linear equations. As an example, for any exponent r, considering

$$g_j(2^r) - g_j(-2^r) = 2^{r+1}a_1 + 2^{3r+1}a_3 + 2^{5r+1}a_5 + \cdots$$

leads to the determination

$$a_1 \equiv \frac{g_j(2^r) - g_j(-2^r)}{2^{r+1}} \pmod{2^{2r+4}\mathbb{Z}_2},$$

and similarly

$$a_2 \equiv \frac{g_j(2^r) + g_j(-2^r) - 2g_j(0)}{2^{2r+1}} \pmod{2^{2r+4}\mathbb{Z}_2}.$$

Analysis of the interpolating functions

For each $j \in \mathbb{Z}$, there is an analytic function g_j such that $g_j(n) = T_{2(k+1)n+j}$ for all $n \in \mathbb{Z}$. Write $g_j(x) = \sum_m a_m x^m$. Then *a priori*

$$\nu_2(a_m) \geq \begin{cases} m + S_2(m) - 1, & k \text{ odd}, \\ m + S_2(m), & k \text{ even}. \end{cases}$$

We will primarily focus on the case where k is odd, since the even k case is similar, but easier. It is immediate that $T_j = g_j(0) = a_0$. In general one may approximate the coefficients a_m by computing $g_j(n)$ for several integers n and solving a system of linear equations. As an example, for any exponent r, considering

$$g_j(2^r) - g_j(-2^r) = 2^{r+1}a_1 + 2^{3r+1}a_3 + 2^{5r+1}a_5 + \cdots$$

leads to the determination

$$a_1 \equiv \frac{g_j(2^r) - g_j(-2^r)}{2^{r+1}} \pmod{2^{2r+4}\mathbb{Z}_2},$$

and similarly

$$a_2 \equiv \frac{g_j(2^r) + g_j(-2^r) - 2g_j(0)}{2^{2r+1}} \pmod{2^{2r+4}\mathbb{Z}_2}.$$

So you can estimate the first few coefficients by knowing the first few terms of the sequence.

For analytic function $g_j(x) = \sum_m a_m x^m$ interpolating $g_j(n) = T_{2(k+1)n+j}$ for $n \in \mathbb{Z}$:

Coefficients of g_j, k odd

j	a_0	a_1
0	0	$8-4k \pmod{2^{k+1-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$1 \le i \le k-1$	1	
k	$k-1$	$0 \pmod{2^{k-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$k+1$	$2k-2$	$4k-8 \pmod{2^{k+1-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
-1	$3-k$	$0 \pmod{2^{k-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$1-k \le i \le -2$	1	
$-k$	-1	

Some series coefficients

For analytic function $g_j(x) = \sum_m a_m x^m$ interpolating $g_j(n) = T_{2(k+1)n+j}$ for $n \in \mathbb{Z}$:

Coefficients of g_j, k odd

j	a_0	a_1
0	0	$8 - 4k \pmod{2^{k+1-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$1 \le i \le k-1$	1	
k	$k-1$	$0 \pmod{2^{k-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$k+1$	$2k-2$	$4k - 8 \pmod{2^{k+1-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
-1	$3-k$	$0 \pmod{2^{k-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$1-k \le i \le -2$	1	
$-k$	-1	

Remark. Note that $a_0 = T_j$. Also observe that, since the functions g_j are analytic on discs in $\mathbb{C}_2$ containing $\mathbb{Z}_2$, that all our results will apply to T_n for *negative* integers n just as well.

Some series coefficients

For analytic function $g_j(x) = \sum_m a_m x^m$ interpolating $g_j(n) = T_{2(k+1)n+j}$ for $n \in \mathbb{Z}$:

Coefficients of g_j, k odd

j	a_0	a_1
0	0	$8-4k \pmod{2^{k+1-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$1 \le i \le k-1$	1	
k	$k-1$	$0 \pmod{2^{k-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$k+1$	$2k-2$	$4k-8 \pmod{2^{k+1-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
-1	$3-k$	$0 \pmod{2^{k-\lfloor \log_2 k \rfloor}\mathbb{Z}_2}$
$1-k \le i \le -2$	1	
$-k$	-1	

Remark. Note that $a_0 = T_j$. Also observe that, since the functions g_j are analytic on discs in $\mathbb{C}_2$ containing $\mathbb{Z}_2$, that all our results will apply to T_n for *negative* integers n just as well.

Remark. Although the functions $g_j(x)$ may be evaluated at rational arguments $x \in \mathbb{Z}_2$, the values obtained do not correspond to values of T_n when $x \notin \mathbb{Z}$. For example, when $k = 5$ the function $g_0(x)$ interpolates the values $\{T_{12x}\}$ when $x \in \mathbb{Z}$ and converges at $x = 1/3 \in \mathbb{Z}_2$, but $g_0(1/3)$ does not equal T_4. The reason for this is that $(\alpha^n)^{1/n}$ does not equal α in general.

Theorem (PTY, 2017)

If the order $k \geq 5$ is odd, then for all integers n, the 2-adic valuation of the generalized Fibonacci number T_n of order k is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, k \pmod{k+1}, \\ \nu_2(k-1), & \text{if } n \equiv k \pmod{2k+2}, \\ \nu_2(k-3), & \text{if } n \equiv -1 \pmod{2k+2}, \\ \nu_2(n-k-1), & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and} \\ & \nu_2(n-k-1) < \nu_2(k^2-1), \\ \nu_2(n-2)+1, & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and} \\ & \nu_2(n-k-1) > \nu_2(k^2-1), \\ \nu_2(n) - \nu_2(k+1) + 1, & \text{if } n \equiv 0 \pmod{2k+2}. \end{cases}$$

Generalized Fibonacci numbers of odd order

Theorem (PTY, 2017)

If the order $k \geq 5$ is odd, then for all integers n, the 2-adic valuation of the generalized Fibonacci number T_n of order k is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, k \pmod{k+1}, \\ \nu_2(k-1), & \text{if } n \equiv k \pmod{2k+2}, \\ \nu_2(k-3), & \text{if } n \equiv -1 \pmod{2k+2}, \\ \nu_2(n-k-1), & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and} \\ & \nu_2(n-k-1) < \nu_2(k^2-1), \\ \nu_2(n-2)+1, & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and} \\ & \nu_2(n-k-1) > \nu_2(k^2-1), \\ \nu_2(n) - \nu_2(k+1) + 1, & \text{if } n \equiv 0 \pmod{2k+2}. \end{cases}$$

This theorem gives the exact valuation $\nu_2(T_n)$ in all cases except when $n \equiv k+1 \pmod{2k+2}$ and $\nu_2(n-k-1) = \nu_2(k^2-1)$. It agrees with what L-M did for $k = 5$.

Generalized Fibonacci numbers of odd order

Theorem (PTY, 2017)

If the order $k \geq 5$ is odd, then for all integers n, the 2-adic valuation of the generalized Fibonacci number T_n of order k is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, k \pmod{k+1}, \\ \nu_2(k-1), & \text{if } n \equiv k \pmod{2k+2}, \\ \nu_2(k-3), & \text{if } n \equiv -1 \pmod{2k+2}, \\ \nu_2(n-k-1), & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and} \\ & \nu_2(n-k-1) < \nu_2(k^2-1), \\ \nu_2(n-2)+1, & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and} \\ & \nu_2(n-k-1) > \nu_2(k^2-1), \\ \nu_2(n) - \nu_2(k+1) + 1, & \text{if } n \equiv 0 \pmod{2k+2}. \end{cases}$$

This theorem gives the exact valuation $\nu_2(T_n)$ in all cases except when $n \equiv k+1 \pmod{2k+2}$ and $\nu_2(n-k-1) = \nu_2(k^2-1)$. It agrees with what L-M did for $k = 5$.

This theorem also implies the odd k case of L-M conjecture that $\nu_2(T_{s(k+1)2^r}) = r + \nu_2(k-2) + 1$ for integers $r \geq 1$ and s odd. That settles that.

Generalized Fibonacci numbers of odd order

Theorem (PTY, 2017)

If the order $k \geq 5$ is odd, then for all integers n, the 2-adic valuation of the generalized Fibonacci number T_n of order k is given by

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \not\equiv 0, k \pmod{k+1}, \\ \nu_2(k-1), & \text{if } n \equiv k \pmod{2k+2}, \\ \nu_2(k-3), & \text{if } n \equiv -1 \pmod{2k+2}, \\ \nu_2(n-k-1), & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and } \\ & \nu_2(n-k-1) < \nu_2(k^2-1), \\ \nu_2(n-2)+1, & \text{if } n \equiv k+1 \pmod{2k+2} \text{ and } \\ & \nu_2(n-k-1) > \nu_2(k^2-1), \\ \nu_2(n)-\nu_2(k+1)+1, & \text{if } n \equiv 0 \pmod{2k+2}. \end{cases}$$

This theorem gives the exact valuation $\nu_2(T_n)$ in all cases except when $n \equiv k+1 \pmod{2k+2}$ and $\nu_2(n-k-1) = \nu_2(k^2-1)$. It agrees with what L-M did for $k = 5$.

This theorem also implies the odd k case of L-M conjecture that $\nu_2(T_{s(k+1)2^r}) = r + \nu_2(k-2) + 1$ for integers $r \geq 1$ and s odd. That settles that.

What about the remaining case? How about the L-M conjecture?

The conjecture goes down!

Theorem

In the case of order $k = 5$, the formula

$$\nu_2(T_n) = \begin{cases} \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) < 8, \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) \geq 8 \end{cases}$$

conjectured by L-M is correct when $\nu_2(n+2) \neq 8$, but is not correct in general.

The conjecture goes down!

Theorem

In the case of order $k = 5$, the formula

$$\nu_2(T_n) = \begin{cases} \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) < 8, \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) \geq 8 \end{cases}$$

conjectured by L-M is correct when $\nu_2(n+2) \neq 8$, but is not correct in general.

The conjecture goes down!

Theorem

In the case of order $k = 5$, the formula

$$\nu_2(T_n) = \begin{cases} \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) < 8, \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) \geq 8 \end{cases}$$

conjectured by L-M is correct when $\nu_2(n+2) \neq 8$, but is not correct in general.

Proof. For the affirmative part, we use the analytic function $g_6(m)$, which interpolates the values T_{12m+6}, to determine T_{12m+6} modulo 2^9. If $\nu_2(m) = 0$, then $\nu_2(T_n) = 2$; if $\nu_2(m) \geq 2$, then $\nu_2(T_n) = 3$; if $\nu_2(m) = 1$, then $T_n \equiv n + 2 \pmod{2^8}$ but not modulo 2^9.

The conjecture goes down!

Theorem

In the case of order $k = 5$, the formula

$$\nu_2(T_n) = \begin{cases} \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) < 8, \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) \geq 8 \end{cases}$$

conjectured by L-M is correct when $\nu_2(n+2) \neq 8$, but is not correct in general.

Proof. For the affirmative part, we use the analytic function $g_6(m)$, which interpolates the values T_{12m+6}, to determine T_{12m+6} modulo 2^9. If $\nu_2(m) = 0$, then $\nu_2(T_n) = 2$; if $\nu_2(m) \geq 2$, then $\nu_2(T_n) = 3$; if $\nu_2(m) = 1$, then $T_n \equiv n + 2 \pmod{2^8}$ but not modulo 2^9.

For the negative part: If you assume the formula were correct for positive integers n, it would necessarily also hold for negative integers n by the continuity of the analytic function $g_6(m)$. However, the formula fails for $n = -43266$, as $\nu_2(T_n) = 20$ while $\nu_2(n + 43266) = +\infty$.

The conjecture goes down!

Theorem

In the case of order $k = 5$, the formula

$$\nu_2(T_n) = \begin{cases} \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) < 8, \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) \geq 8 \end{cases}$$

conjectured by L-M is correct when $\nu_2(n+2) \neq 8$, but is not correct in general.

Proof. For the affirmative part, we use the analytic function $g_6(m)$, which interpolates the values T_{12m+6}, to determine T_{12m+6} modulo 2^9. If $\nu_2(m) = 0$, then $\nu_2(T_n) = 2$; if $\nu_2(m) \geq 2$, then $\nu_2(T_n) = 3$; if $\nu_2(m) = 1$, then $T_n \equiv n + 2 \pmod{2^8}$ but not modulo 2^9.

For the negative part: If you assume the formula were correct for positive integers n, it would necessarily also hold for negative integers n by the continuity of the analytic function $g_6(m)$. However, the formula fails for $n = -43266$, as $\nu_2(T_n) = 20$ while $\nu_2(n + 43266) = +\infty$.

The first two positive integer counterexamples are $n = 3 \cdot 2^{20} - 43266$, with $\nu_2(T_n) = 22$ while $\nu_2(n + 43266) = 20$; and $n = 3 \cdot 2^{21} - 43266$, with $\nu_2(T_n) = 20$ while $\nu_2(n + 43266) = 21$.

The conjecture goes down!

Theorem

In the case of order $k = 5$, the formula

$$\nu_2(T_n) = \begin{cases} \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) < 8, \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) \geq 8 \end{cases}$$

conjectured by L-M is correct when $\nu_2(n+2) \neq 8$, but is not correct in general.

Proof. For the affirmative part, we use the analytic function $g_6(m)$, which interpolates the values T_{12m+6}, to determine T_{12m+6} modulo 2^9. If $\nu_2(m) = 0$, then $\nu_2(T_n) = 2$; if $\nu_2(m) \geq 2$, then $\nu_2(T_n) = 3$; if $\nu_2(m) = 1$, then $T_n \equiv n + 2 \pmod{2^8}$ but not modulo 2^9.

For the negative part: If you assume the formula were correct for positive integers n, it would necessarily also hold for negative integers n by the continuity of the analytic function $g_6(m)$. However, the formula fails for $n = -43266$, as $\nu_2(T_n) = 20$ while $\nu_2(n + 43266) = +\infty$.

The first two positive integer counterexamples are $n = 3 \cdot 2^{20} - 43266$, with $\nu_2(T_n) = 22$ while $\nu_2(n + 43266) = 20$; and $n = 3 \cdot 2^{21} - 43266$, with $\nu_2(T_n) = 20$ while $\nu_2(n + 43266) = 21$.

It's a shame that the conjecture fails, after deceiving us for three million (positive) terms.

We can fix the conjecture

So it turns out that 43266 is not the magic number. However, we can say this:

Theorem (Repaired Conjecture)

In the case of order $k = 5$, *there exists some 2-adic integer* $y \in \mathbb{Z}_2$ *which satisfies* $y \equiv 43266 - 3 \cdot 2^{20} \pmod{2^{22}\mathbb{Z}_2}$, *such that*

$$\nu_2(T_n) = \nu_2(n + y)$$

for all integers $n \equiv 6 \pmod{12}$.

We can fix the conjecture

So it turns out that 43266 is not the magic number. However, we can say this:

Theorem (Repaired Conjecture)

In the case of order $k = 5$, there exists some 2-adic integer $y \in \mathbb{Z}_2$ which satisfies $y \equiv 43266 - 3 \cdot 2^{20} \pmod{2^{22}\mathbb{Z}_2}$, such that

$$\nu_2(T_n) = \nu_2(n + y)$$

for all integers $n \equiv 6 \pmod{12}$.

We know that y exists by considering the Newton polygon of $g_6(x)$; in fact $-y = 12z + 6$ where z is a zero of the function $g_6(x)$ (which is a linear combination of exponentials).

We can fix the conjecture

So it turns out that 43266 is not the magic number. However, we can say this:

Theorem (Repaired Conjecture)

In the case of order $k = 5$, there exists some 2-adic integer $y \in \mathbb{Z}_2$ which satisfies $y \equiv 43266 - 3 \cdot 2^{20} \pmod{2^{22}\mathbb{Z}_2}$, such that

$$\nu_2(T_n) = \nu_2(n + y)$$

for all integers $n \equiv 6 \pmod{12}$.

We know that y exists by considering the Newton polygon of $g_6(x)$; in fact $-y = 12z + 6$ where z is a zero of the function $g_6(x)$ (which is a linear combination of exponentials).

Note that if y were a positive integer, it would mean that $T_{-y} = 0$.

We can fix the conjecture

So it turns out that 43266 is not the magic number. However, we can say this:

Theorem (Repaired Conjecture)

In the case of order $k = 5$, there exists some 2-adic integer $y \in \mathbb{Z}_2$ which satisfies $y \equiv 43266 - 3 \cdot 2^{20} \pmod{2^{22}\mathbb{Z}_2}$, such that

$$\nu_2(T_n) = \nu_2(n + y)$$

for all integers $n \equiv 6 \pmod{12}$.

We know that y exists by considering the Newton polygon of $g_6(x)$; in fact $-y = 12z + 6$ where z is a zero of the function $g_6(x)$ (which is a linear combination of exponentials).

Note that if y were a positive integer, it would mean that $T_{-y} = 0$.

Theorem

Suppose $k \geq 5$ is odd and let $a = \nu_2(k-1)$. Then there exists a 2-adic integer $z \in \mathbb{Z}_2$ with $\nu_2(z) = a - 1$ and

$$z \equiv \frac{k-1}{4-2k} \pmod{2^{3a-1}\mathbb{Z}_2}$$

such that $\nu_2(T_n) = \nu_2(m - z) + 2$ when n is of the form $n = (2k+2)m + k + 1$.

What is the nature of the "rogue zero" z?

It seems heuristically sensible to suggest the following conjecture: For all odd orders $k \geq 5$, the 2-adic integer z which satisfies $\nu_2(T_n) = \nu_2(m - z) + 2$ when n is of the form $n = (2k + 2)m + k + 1$ is always transcendental. Or at least it's never an integer.

What is the nature of the "rogue zero" z?

It seems heuristically sensible to suggest the following conjecture: For all odd orders $k \geq 5$, the 2-adic integer z which satisfies $\nu_2(T_n) = \nu_2(m - z) + 2$ when n is of the form $n = (2k + 2)m + k + 1$ is always transcendental. Or at least it's never an integer.

This z corresponds to a zero $x = (2k + 2)z + k + 1$ of $c_1\alpha_1^x + \cdots + c_k\alpha_k^x = 0$.

What is the nature of the "rogue zero" z?

It seems heuristically sensible to suggest the following conjecture: For all odd orders $k \geq 5$, the 2-adic integer z which satisfies $\nu_2(T_n) = \nu_2(m - z) + 2$ when n is of the form $n = (2k+2)m + k + 1$ is always transcendental. Or at least it's never an integer.

This z corresponds to a zero $x = (2k+2)z + k + 1$ of $c_1\alpha_1^x + \cdots + c_k\alpha_k^x = 0$.

Knowing the nature of z seems to be related to Diophantine equations concerning (T_n).

What is the nature of the "rogue zero" z?

It seems heuristically sensible to suggest the following conjecture: For all odd orders $k \geq 5$, the 2-adic integer z which satisfies $\nu_2(T_n) = \nu_2(m - z) + 2$ when n is of the form $n = (2k+2)m + k + 1$ is always transcendental. Or at least it's never an integer.

This z corresponds to a zero $x = (2k+2)z + k + 1$ of $c_1\alpha_1^x + \cdots + c_k\alpha_k^x = 0$.

Knowing the nature of z seems to be related to Diophantine equations concerning (T_n).

Whether z is an integer is connected to the nonvanishing of (T_{-n}). The sequences (T_{-n}) begin like this:

$k = 2$: 1, -1, 2, -3, 5, -8, 13, -21, 34, -55, 89, -144, 233, -377, 610, -987, 1597, -2584, . . .

$k = 3$: 0, 1, -1, 0, 2, -3, 1, 4, -8, 5, 7, -20, 18, 9, -47, 56, 0, -103, 159, -56, . . .

$k = 4$: -1, 1, 1, -1, -2, 3, 1, -3, -3, 8, -1, -7, -3, 19, -10, -13, 1, 41, -39, -16, . . .

$k = 5$: -2, 1, 1, 1, -1, -4, 4, 1, 1, -3, -7, 12, -2, 1, -7, -11, 31, -16, 4, -15, . . .

$k = 6$: -3, 1, 1, 1, 1, -1, -6, 5, 1, 1, 1, -3, -11, 16, -3, 1, 1, -7, -19, 43, . . .

It seems heuristically sensible to suggest the following conjecture: For all odd orders $k \geq 5$, the 2-adic integer z which satisfies $\nu_2(T_n) = \nu_2(m - z) + 2$ when n is of the form $n = (2k+2)m + k + 1$ is always transcendental. Or at least it's never an integer.

This z corresponds to a zero $x = (2k+2)z + k + 1$ of $c_1\alpha_1^x + \cdots + c_k\alpha_k^x = 0$.

Knowing the nature of z seems to be related to Diophantine equations concerning (T_n).

Whether z is an integer is connected to the nonvanishing of (T_{-n}). The sequences (T_{-n}) begin like this:

$k = 2$: 1, -1, 2, -3, 5, -8, 13, -21, 34, -55, 89, -144, 233, -377, 610, -987, 1597, -2584, . . .

$k = 3$: 0, 1, -1, 0, 2, -3, 1, 4, -8, 5, 7, -20, 18, 9, -47, 56, 0, -103, 159, -56, . . .

$k = 4$: -1, 1, 1, -1, -2, 3, 1, -3, -3, 8, -1, -7, -3, 19, -10, -13, 1, 41, -39, -16, . . .

$k = 5$: -2, 1, 1, 1, -1, -4, 4, 1, 1, -3, -7, 12, -2, 1, -7, -11, 31, -16, 4, -15, . . .

$k = 6$: -3, 1, 1, 1, 1, -1, -6, 5, 1, 1, 1, -3, -11, 16, -3, 1, 1, -7, -19, 43, . . .

One may also conjecture that the values $T_{-1} = T_{-4} = T_{-17} = 0$ when $k = 3$ are the only solutions to $T_{-n} = 0$ for any order k. Sobolewski's result shows that there are no such solutions for k even. For any given odd $k \geq 5$, there is *at most one* solution to $T_{-n} = 0$.

What is the nature of the "rogue zero" z?

It seems heuristically sensible to suggest the following conjecture: For all odd orders $k \geq 5$, the 2-adic integer z which satisfies $\nu_2(T_n) = \nu_2(m - z) + 2$ when n is of the form $n = (2k+2)m + k + 1$ is always transcendental. Or at least it's never an integer.

This z corresponds to a zero $x = (2k+2)z + k + 1$ of $c_1\alpha_1^x + \cdots + c_k\alpha_k^x = 0$.

Knowing the nature of z seems to be related to Diophantine equations concerning (T_n).

Whether z is an integer is connected to the nonvanishing of (T_{-n}). The sequences (T_{-n}) begin like this:

$k = 2$: 1, -1, 2, -3, 5, -8, 13, -21, 34, -55, 89, -144, 233, -377, 610, -987, 1597, -2584, . . .

$k = 3$: 0, 1, -1, 0, 2, -3, 1, 4, -8, 5, 7, -20, 18, 9, -47, 56, 0, -103, 159, -56, . . .

$k = 4$: -1, 1, 1, -1, -2, 3, 1, -3, -3, 8, -1, -7, -3, 19, -10, -13, 1, 41, -39, -16, . . .

$k = 5$: -2, 1, 1, 1, -1, -4, 4, 1, 1, -3, -7, 12, -2, 1, -7, -11, 31, -16, 4, -15, . . .

$k = 6$: -3, 1, 1, 1, 1, -1, -6, 5, 1, 1, 1, -3, -11, 16, -3, 1, 1, -7, -19, 43, . . .

One may also conjecture that the values $T_{-1} = T_{-4} = T_{-17} = 0$ when $k = 3$ are the only solutions to $T_{-n} = 0$ for any order k. Sobolewski's result shows that there are no such solutions for k even. For any given odd $k \geq 5$, there is *at most one* solution to $T_{-n} = 0$.

It seems to also be interesting to consider the 2-adic valuation of $T_n \pm 1$.

Thank You!

Some values of T_n

$$
\begin{aligned}
T_{-2k-2} &= 4k-8 \quad (\text{if} \quad k \geq 3) \\
T_{-2k-1} &= 13-4k \\
T_{-2k} &= -3 \\
T_{-k-i} &= 1 \quad \text{for} \quad 3 \leq i \leq k-1 \\
T_{-k-2} &= k-1 \quad (\text{if} \quad k \geq 3) \\
T_{-k-1} &= 6-2k \\
T_{-k} &= -1 \\
T_{-i} &= 1 \quad \text{for} \quad 2 \leq i \leq k-1 \\
T_{-1} &= 3-k \\
T_0 &= 0 \\
T_i &= 1 \quad \text{for} \quad 1 \leq i \leq k-1 \\
T_k &= k-1 \\
T_{k+i} &= 2^{i-1}(2k-3)+1 \quad \text{for} \quad 1 \leq i \leq k \\
T_{2k+1} &= 2^k(2k-3)-k+3 \\
T_{2k+2} &= 2^{k+1}(2k-3)-4k+8
\end{aligned}
$$