Stability for Take-Away Games

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α-take away game, or α-TAG

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- One pile of stones, initially n stones
- $\alpha \geq 1$
- Two players alternate moves
- First turn: take at least one stone, but not all of them
- Subsequent turns: take up to α times as many stones as last player took
- Goal: Take the last stone. (More precisely, if it's your turn and you can't move, you lose.)

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- $\alpha = 1$: $\mathcal{P}$ positions are powers of 2, together with 0
- $\alpha = 2$: $\mathcal{P}$ positions are Fibonacci numbers

$\mathcal{P}$ positions for α-TAG

Theorem (Schwenk)

The $\mathcal{P}$ positions satisfy the following recurrence:

$$P_{n+1} = P_n + P_m,$$

where m is the unique integer such that $\alpha P_{m-1} < P_n \leq \alpha P_m$.

Proof relies on a generalization of Zeckendorf's theorem.

Zeckendorf's Theorem

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Zeckendorf representation is constructed greedily, by choosing the largest Fibonacci number possible from the remainder.

Example

$40 = 34 + 5 + 1.$

Generalization of Zeckendorf's Theorem

Theorem (Generalized Zeckendorf's Theorem)

Let $\alpha \geq 1$, and let P_n be a sequence defined by the recurrence $P_{n+1} = P_n + P_m$, where m is the unique integer such that $\alpha P_{m-1} < P_n \leq \alpha P_m$, with initial conditions $P_0 = 0$ and $P_1 = 1$. Then every positive integer n can be expressed uniquely in the form $n = P_{i_1} + P_{i_2} + \cdots + P_{i_k}$, where $\alpha P_{i_j} < P_{i_{j+1}}$.

Construction here is also greedy: take the largest P_n you can.

Winning strategy for α-TAG

Suppose there are n stones. Write $n = P_{i_1} + P_{i_2} + \cdots + P_{i_k}$ as in the generalized Zeckendorf theorem. Then, on every move, remove the smallest generalized Zeckendorf part P_{i_1}. The next player will not be able to do so.

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Example

$\alpha = 3$, $n = 35$.

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$\alpha = 3$, $n = 35$. $\mathcal{P}$-positions are 0, 1, 2, 3, 4, 6, 8, 11, 15, 21, 29, 37, ...
Generalized Zeckendorf representation: $35 = 29 + 6$.

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Sample game play with good play by first player:
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Sample game play with good play by first player:
$35 \to 29$

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Sample game play with good play by first player:
$35 \to 29 \to 28$

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$35 \to 29 \to 28 \to 27$

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Sample game play with good play by first player:
$35 \to 29 \to 28 \to 27 \to 25$

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$35 \to 29 \to 28 \to 27 \to 25 \to 21 \to 13 \to 11 \to 7 \to 6 \to 4$

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Sample game play with good play by first player:
$35 \to 29 \to 28 \to 27 \to 25 \to 21 \to 13 \to 11 \to 7 \to 6 \to 4 \to 0$.

Stability Theorem

Theorem

For every $\alpha > 1$, there exists a half-open interval $I_\alpha = [a_\alpha, b_\alpha)$ containing α such that for all $\beta \in I_\alpha$, the $\mathcal{P}$ positions of α-TAG are the same as those of β-TAG.

Eventual recurrence

Theorem

For any $\alpha \geq 1$, there exist integers $k, N \geq 0$ such that for all $n \geq N$, $P_{n+1} = P_n + P_{n-k}$.

However, first few terms might not satisfy this recurrence.

Stable intervals

Range	Eventual recurrence	Initial conditions
$1 \leq \alpha < 2$	$P_{n+1} = P_n + P_n$	0,1
$2 \leq \alpha < \frac{5}{2}$	$P_{n+1} = P_n + P_{n-1}$	0,1,2
$\frac{5}{2} \leq \alpha < 3$	$P_{n+1} = P_n + P_{n-2}$	0,1,2,3,5
$3 \leq \alpha < \frac{7}{2}$	$P_{n+1} = P_n + P_{n-3}$	0,1,2,3,4,6
$\frac{7}{2} \leq \alpha < \frac{11}{3}$	$P_{n+1} = P_n + P_{n-4}$	0,1,2,3,4,6,8,11,15,21
$\frac{11}{3} \leq \alpha < \frac{43}{11}$	$P_{n+1} = P_n + P_{n-4}$	0,1,2,3,4,6,8,11
$\frac{43}{11} \leq \alpha < 4$	$P_{n+1} = P_n + P_{n-5}$	0,1,2,3,4,6,8,11,14,18,24,32,43
$4 \leq \alpha < \frac{13}{3}$	$P_{n+1} = P_n + P_{n-5}$	0,1,2,3,4,5,7,9,12
$\frac{13}{3} \leq \alpha < \frac{31}{7}$	$P_{n+1} = P_n + P_{n-6}$	0,1,2,3,4,5,7,9,12,15,19,24,31,40,52
$\frac{31}{7} \leq \alpha < \frac{9}{2}$	$P_{n+1} = P_n + P_{n-6}$	0,1,2,3,4,5,7,9,12,15,19,24,31
$\frac{9}{2} \leq \alpha < \frac{14}{3}$	$P_{n+1} = P_n + P_{n-6}$	0,1,2,3,4,5,7,9,11,14,18

Cutoffs

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The first few cutoffs are $2, \frac{5}{2}, 3, \frac{7}{2}, \frac{11}{3}, \frac{43}{11}, 4, \frac{13}{3}, \frac{31}{7}, \frac{9}{2}, \frac{14}{3}$.

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First $\mathcal{P}$ positions are 0, 1, 2, 3, 4, 6, 8, 11, 15, 21, 29, 40, 55, ...

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For example, we form 21 as $15 + 6$, because $\alpha \cdot 4 < 15 \leq \alpha \cdot 6$ with $\alpha = 3$.

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Similarly, we compare other terms, to find that the next cutoff is

$$\min\left\{\frac{4}{1}, \frac{6}{1}, \frac{8}{2}, \frac{11}{3}, \frac{15}{4}, \frac{21}{6}, \frac{29}{8}, \frac{40}{11}, \ldots\right\} = \frac{21}{6} = \frac{7}{2}.$$

Theorems and questions about cutoffs

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Conjecture

For every positive integer d, there exists an integer N such that for all $k > N$, $\frac{k}{d}$ is a cutoff.

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$C(x)$: number of cutoffs $\leq x$. Appears that $C(x) = \rho x^2 + o(x^2)$ for some ρ. Probably $\rho \approx 2$.

Thank you

Thank you for your attention!