Counting subgroups of the multiplicative group

Lee Troupe
joint w/ Greg Martin

University of British Columbia

West Coast Number Theory
2017

Question from I. Shparlinski to G. Martin, circa 2009:

How many subgroups does $\mathbb{Z}_n^\times := (\mathbb{Z}/n\mathbb{Z})^\times$ *usually have?*

Question from I. Shparlinski to G. Martin, circa 2009:

How many subgroups does $\mathbb{Z}_n^\times := (\mathbb{Z}/n\mathbb{Z})^\times$ *usually have?*

Let $I(n)$ denote the number of isomorphism classes of subgroups of $\mathbb{Z}_n^\times$.
Let $G(n)$ denote the number of subsets of $\mathbb{Z}_n^\times$ which are subgroups.

Shparlinski's question concerns the distribution of values of $I(n)$ and/or $G(n)$.

To set the stage: What do we talk about when we talk about distributions of arithmetic functions?

Average order

Let $f(n)$ be an arithmetic function.

We can ask for the *average order* of $f(n)$, i.e. a function $g(n)$ so that

$$\frac{1}{x}\sum_{n\leq x} f(n) \sim g(n).$$

Average order

Let $f(n)$ be an arithmetic function.

We can ask for the *average order* of $f(n)$, i.e. a function $g(n)$ so that

$$\frac{1}{x}\sum_{n\leq x} f(n) \sim g(n).$$

Example: The average order of the number-of-prime-factors function $\omega(n)$ is $\log\log n$ (proof: insert the definition of $\omega(n)$, swap the order of summation, use Mertens's theorem).

This could be a starting point for studying $I(n)$ and $G(n)$, but it doesn't really answer the question.

Normal order

We can ask for the *normal order* of $f(n)$, i.e. a function $g(n)$ so that, for any $\epsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \left| \frac{f(n)}{g(n)} - 1 \right| < \epsilon \right\} = 1.$$

Normal order

We can ask for the *normal order* of $f(n)$, i.e. a function $g(n)$ so that, for any $\epsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x : \left| \frac{f(n)}{g(n)} - 1 \right| < \epsilon \right\} = 1.$$

Theorem (Hardy, Ramanujan 1917)

The normal order of $\omega(n)$ *is* $\log\log n$.

Turán (1934): Proof via an upper bound for the variance (second moment) of the form

$$\frac{1}{x} \sum_{n\leq x} (\omega(n) - \log\log n)^2 \ll \log\log x.$$

We can ask for more.

The fundamental theorem of probabilistic number theory

Theorem (Erdős, Kac 1940)

Let $\omega(n)$ denote the number of distinct prime factors of a number n. Then

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} < u \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt.$$

In other words, the values of the function $\omega(n)$ are normally distributed, with mean and variance both equal to $\log\log n$.

The fundamental theorem of probabilistic number theory

Theorem (Erdős, Kac 1940)

Let $\omega(n)$ denote the number of distinct prime factors of a number n. Then

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} < u \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt.$$

In other words, the values of the function $\omega(n)$ are normally distributed, with mean and variance both equal to $\log\log n$.

Halberstam (1954): Proof by the method of moments, i.e. finding asymptotic formulas for each of the central moments

$$\sum_{n\le x} (\omega(n) - \log\log n)^k.$$

The fundamental theorem of probabilistic number theory

Erdős and Kac's paper establishes a normal-distribution result for a wide class of *additive functions* $f(n)$: $f(p_1^{e_1} \cdots p_k^{e_k}) = f(p_1^{e_1}) + \cdots + f(p_k^{e_k})$.

The fundamental theorem of probabilistic number theory

Erdős and Kac's paper establishes a normal-distribution result for a wide class of *additive functions* $f(n)$: $f(p_1^{e_1} \cdots p_k^{e_k}) = f(p_1^{e_1}) + \cdots + f(p_k^{e_k})$.

Definition

We say a function $f(n)$ *satisfies an Erdős–Kac law with mean* $\mu(n)$ *and variance* $\sigma^2(n)$ if

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \le x : \frac{f(n) - \mu(n)}{\sigma(n)} < u \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2} \, dt.$$

Erdős–Kac laws for non-additive functions

Theorem (Liu 2006)

For any elliptic curve $E/\mathbb{Q}$ with CM, $\omega(\#E(\mathbb{F}_p))$ satisfies an Erdős–Kac law with mean and variance $\log\log p$.

Erdős–Kac laws for non-additive functions

Theorem (Liu 2006)

For any elliptic curve $E/\mathbb{Q}$ with CM, $\omega(\#E(\mathbb{F}_p))$ satisfies an Erdős–Kac law with mean and variance $\log\log p$.

Theorem (Erdős, Pomerance 1985)

The functions $\omega(\varphi(n))$ and $\Omega(\varphi(n))$ both satisfy an Erdős–Kac law, with mean $\frac{1}{2}(\log\log n)^2$ and variance $\frac{1}{3}(\log\log n)^3$.

Erdős–Kac laws for non-additive functions

Theorem (Liu 2006)

For any elliptic curve $E/\mathbb{Q}$ with CM, $\omega(\#E(\mathbb{F}_p))$ satisfies an Erdős–Kac law with mean and variance $\log\log p$.

Theorem (Erdős, Pomerance 1985)

The functions $\omega(\varphi(n))$ and $\Omega(\varphi(n))$ both satisfy an Erdős–Kac law, with mean $\frac{1}{2}(\log\log n)^2$ and variance $\frac{1}{3}(\log\log n)^3$.

$\Omega(\varphi(n))$ is additive;

Erdős–Kac laws for non-additive functions

Theorem (Liu 2006)

For any elliptic curve $E/\mathbb{Q}$ with CM, $\omega(\#E(\mathbb{F}_p))$ satisfies an Erdős–Kac law with mean and variance $\log\log p$.

Theorem (Erdős, Pomerance 1985)

The functions $\omega(\varphi(n))$ and $\Omega(\varphi(n))$ both satisfy an Erdős–Kac law, with mean $\frac{1}{2}(\log\log n)^2$ and variance $\frac{1}{3}(\log\log n)^3$.

$\Omega(\varphi(n))$ is additive; $\omega(\varphi(n))$ isn't!

Both are φ-*additive*: If $\varphi(n) = p_1^{e_1} \cdots p_k^{e_k}$, then

$$f(\varphi(n)) = f(p_1^{e_1}) + \cdots + f(p_k^{e_k}).$$

φ-additivity

Recall: $I(n)$ is the number of ismorphism classes of subgroups of $\mathbb{Z}_n^\times$.
$G(n)$ is the number of subsets of $\mathbb{Z}_n^\times$ which are subgroups.

Fact: Every finite abelian group is the direct product of its Sylow p-subgroups.

φ-additivity

Recall: $I(n)$ is the number of ismorphism classes of subgroups of $\mathbb{Z}_n^\times$.
$G(n)$ is the number of subsets of $\mathbb{Z}_n^\times$ which are subgroups.

Fact: Every finite abelian group is the direct product of its Sylow p-subgroups.

So if $G_p(n)$ denotes the number of subgroups of the Sylow p-subgroup of $\mathbb{Z}_n^\times$, then

$$G(n) = \prod_{p|\varphi(n)} G_p(n) \implies \log G(n) = \sum_{p|\varphi(n)} \log G_p(n)$$

and similarly for $\log I(n)$.

Thus, $\log G(n)$ and $\log I(n)$ are φ-additive functions, as well.

Erdős–Kac laws for subgroups of $\mathbb{Z}_n^\times$

Theorem (Martin–T., submitted)

The function $\log I(n)$ *satisfies an Erdős–Kac law with mean* $\frac{\log 2}{2}(\log\log n)^2$ *and variance* $\frac{\log 2}{3}(\log\log n)^3$.

Erdős–Kac laws for subgroups of $\mathbb{Z}_n^\times$

Theorem (Martin–T., submitted)

The function $\log I(n)$ *satisfies an Erdős–Kac law with mean* $\frac{\log 2}{2}(\log\log n)^2$ *and variance* $\frac{\log 2}{3}(\log\log n)^3$.

Theorem (Martin–T., submitted)

The function $\log G(n)$ *satisfies an Erdős–Kac law with mean* $A(\log\log n)^2$ *and variance* $C(\log\log n)^3$.

Erdős–Kac laws for subgroups of $\mathbb{Z}_n^\times$

Theorem (Martin–T., submitted)

The function $\log I(n)$ *satisfies an Erdős–Kac law with mean* $\frac{\log 2}{2}(\log\log n)^2$ *and variance* $\frac{\log 2}{3}(\log\log n)^3$.

Theorem (Martin–T., submitted)

The function $\log G(n)$ *satisfies an Erdős–Kac law with mean* $A(\log\log n)^2$ *and variance* $C(\log\log n)^3$.

It turns out that $A \approx 0.72109$, while $\frac{\log 2}{2} \approx 0.34657$. So, typically, $G(n) \approx I(n)^{2.08}$.

Erdős–Kac laws for subgroups of $\mathbb{Z}_n^\times$

Theorem (Martin–T., submitted)

The function $\log G(n)$ *satisfies an Erdős–Kac law with mean* $A(\log\log n)^2$ *and variance* $C(\log\log n)^3$.

- $A_0 = \dfrac{1}{4}\displaystyle\sum_p \dfrac{p^2\log p}{(p-1)^3(p+1)}$
- $A = A_0 + \dfrac{\log 2}{2} \approx 0.72109$
- $B = \dfrac{1}{4}\displaystyle\sum_p \dfrac{p^3(p^4-p^3-p^2-p-1)(\log p)^2}{(p-1)^6(p+1)^2(p^2+p+1)}$
- $C = \dfrac{(\log 2)^2}{3} + 2A_0\log 2 + 4A_0^2 + B \approx 3.924$

$\mathbb{Z}_n^\times$ with many subgroups

Theorem (Martin–T., submitted)

- *The order of magnitude of the maximal order of* $\log I(n)$ *is* $\log x / \log\log x$. *More precisely,*

$$\frac{\log 2}{5}\frac{\log x}{\log\log x} \lesssim \max_{n \le x} \log I(n) \lesssim \pi\sqrt{\frac{2}{3}}\frac{\log x}{\log\log x}.$$

$\mathbb{Z}_n^\times$ with many subgroups

Theorem (Martin–T., submitted)

- *The order of magnitude of the maximal order of* $\log I(n)$ *is* $\log x/\log\log x$. *More precisely,*

$$\frac{\log 2}{5}\frac{\log x}{\log\log x} \lesssim \max_{n\le x}\log I(n) \lesssim \pi\sqrt{\frac{2}{3}}\frac{\log x}{\log\log x}.$$

- *The order of magnitude of the maximal order of* $\log G(n)$ *is* $(\log x)^2/\log\log x$. *More precisely,*

$$\frac{1}{16}\frac{(\log x)^2}{\log\log x} \lesssim \max_{n\le x}\log G(n) \lesssim \frac{1}{4}\frac{(\log x)^2}{\log\log x}.$$

$\mathbb{Z}_n^\times$ with many subgroups

Theorem (Martin–T., submitted)

- *The order of magnitude of the maximal order of* $\log I(n)$ *is* $\log x / \log\log x$. *More precisely,*

$$\frac{\log 2}{5} \frac{\log x}{\log\log x} \lesssim \max_{n \le x} \log I(n) \lesssim \pi \sqrt{\frac{2}{3}} \frac{\log x}{\log\log x}.$$

- *The order of magnitude of the maximal order of* $\log G(n)$ *is* $(\log x)^2 / \log\log x$. *More precisely,*

$$\frac{1}{16} \frac{(\log x)^2}{\log\log x} \lesssim \max_{n \le x} \log G(n) \lesssim \frac{1}{4} \frac{(\log x)^2}{\log\log x}.$$

Corollary

For any $A > 0$*, there are infinitely many* n *such that* $G(n) > n^A$.

Proof of Erdős–Kac for $\log I(n)$

Recall: Since every subgroup of $\mathbb{Z}_n^\times$ is a direct product of subgroups of the Sylow p-subgroups of $\mathbb{Z}_n^\times$,

$$\log I(n) = \sum_{p \mid \varphi(n)} \log I_p(n).$$

Proof of Erdős–Kac for $\log I(n)$

Recall: Since every subgroup of $\mathbb{Z}_n^\times$ is a direct product of subgroups of the Sylow p-subgroups of $\mathbb{Z}_n^\times$,

$$\log I(n) = \sum_{p \mid \varphi(n)} \log I_p(n).$$

For all $p \mid \varphi(n)$, each $I_p(n)$ counts the trivial subgroup and the entire Sylow p-subgroup of $\mathbb{Z}_n^\times$, and so each $I_p(n) \geq 2$. So

$$\omega(\varphi(n)) \log 2 \leq \log I(n).$$

Proof of Erdős–Kac for $\log l(n)$

For an upper bound: Write the Sylow p-subgroup of $\mathbb{Z}_n^\times$ as

$$\mathbb{Z}_{p^\alpha} := \mathbb{Z}_{p^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p^{\alpha_m}}$$

for some partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $\mathrm{ord}_p(\varphi(n))$.

Proof of Erdős–Kac for $\log I(n)$

For an upper bound: Write the Sylow p-subgroup of $\mathbb{Z}_n^\times$ as

$$\mathbb{Z}_{p^\alpha} := \mathbb{Z}_{p^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p^{\alpha_m}}$$

for some partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $\mathrm{ord}_p(\varphi(n))$.

There is a one-to-one correspondence between subgroups of $\mathbb{Z}_{p^\alpha}$ and subpartitions of α.

Proof of Erdős–Kac for $\log I(n)$

For an upper bound: Write the Sylow p-subgroup of $\mathbb{Z}_n^\times$ as

$$\mathbb{Z}_{p^\alpha} := \mathbb{Z}_{p^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p^{\alpha_m}}$$

for some partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $\mathrm{ord}_p(\varphi(n))$.

There is a one-to-one correspondence between subgroups of $\mathbb{Z}_{p^\alpha}$ and subpartitions of α. Now,

$$\#\{\text{subpartitions of } \alpha\} \leq 2^{\mathrm{ord}_p(\varphi(n))}.$$

Proof of Erdős–Kac for $\log I(n)$

For an upper bound: Write the Sylow p-subgroup of $\mathbb{Z}_n^\times$ as

$$\mathbb{Z}_{p^\alpha} := \mathbb{Z}_{p^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p^{\alpha_m}}$$

for some partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $\mathrm{ord}_p(\varphi(n))$.

There is a one-to-one correspondence between subgroups of $\mathbb{Z}_{p^\alpha}$ and subpartitions of α. Now,

$$\#\{\text{subpartitions of } \alpha\} \le 2^{\mathrm{ord}_p(\varphi(n))}.$$

Therefore

$$\log I(n) \le \sum_{p \mid \varphi(n)} \mathrm{ord}_p(\varphi(n)) \log 2 = \Omega(\varphi(n)) \log 2$$

Proof of Erdős–Kac for $\log I(n)$

For an upper bound: Write the Sylow p-subgroup of $\mathbb{Z}_n^\times$ as

$$\mathbb{Z}_{p^\alpha} := \mathbb{Z}_{p^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p^{\alpha_m}}$$

for some partition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $\mathrm{ord}_p(\varphi(n))$.

There is a one-to-one correspondence between subgroups of $\mathbb{Z}_{p^\alpha}$ and subpartitions of α. Now,

$$\#\{\text{subpartitions of } \alpha\} \leq 2^{\mathrm{ord}_p(\varphi(n))}.$$

Therefore

$$\log I(n) \leq \sum_{p|\varphi(n)} \mathrm{ord}_p(\varphi(n)) \log 2 = \Omega(\varphi(n)) \log 2$$

$$\implies \omega(\varphi(n)) \log 2 \leq \log I(n) \leq \Omega(\varphi(n)) \log 2. \quad \square$$

What about $\log G(n)$?

Given a subpartition $\beta \prec \alpha$, let $N_p(\alpha, \beta)$ be the number of subgroups of $\mathbb{Z}_{p^\alpha}$ isomorphic to $\mathbb{Z}_{p^\beta}$.

Lemma (immediate)

$$\log G_p(n) = \sum_{\beta \prec \alpha} \log N_p(\alpha, \beta).$$

What about $\log G(n)$?

Given a subpartition $\beta \prec \alpha$, let $N_p(\alpha, \beta)$ be the number of subgroups of $\mathbb{Z}_{p^\alpha}$ isomorphic to $\mathbb{Z}_{p^\beta}$.

Lemma (immediate)

$$\log G_p(n) = \sum_{\beta \prec \alpha} \log N_p(\alpha, \beta).$$

Let $\mathbf{b} = (b_1, \ldots, b_{\beta_1})$ and $\mathbf{a} = (a_1, \ldots, a_{\alpha_1})$ be the partitions conjugate to β and α respectively.

What about $\log G(n)$?

Given a subpartition $\beta \prec \alpha$, let $N_p(\alpha, \beta)$ be the number of subgroups of $\mathbb{Z}_{p^\alpha}$ isomorphic to $\mathbb{Z}_{p^\beta}$.

Lemma (immediate)

$$\log G_p(n) = \sum_{\beta \prec \alpha} \log N_p(\alpha, \beta).$$

Let $\mathbf{b} = (b_1, \ldots, b_{\beta_1})$ and $\mathbf{a} = (a_1, \ldots, a_{\alpha_1})$ be the partitions conjugate to β and α respectively. One definition of "conjugate partition": a_j is the number of parts of α of size at least j.

Subgroups and partitions

It turns out that

$$N_p(\alpha, \beta) \approx \prod_{j=1}^{\alpha_1} p^{(a_j - b_j) b_j}.$$

Subgroups and partitions

It turns out that

$$N_p(\alpha, \beta) \approx \prod_{j=1}^{\alpha_1} p^{(a_j - b_j)b_j}.$$

As a function of b_j, the maximum of $(a_j - b_j)b_j$ occurs at $b_j = a_j/2$. With this choice, $p^{(a_j-b_j)b_j} = p^{a_j^2/4}$. These values, corresponding to the choice "$\beta = \frac{1}{2}\alpha$,", provide the largest value of $N_p(\alpha, \beta)$.

Subgroups and partitions

Lemma

For any prime $p \mid \varphi(n)$,

$$\log G_p(n) = \frac{\log p}{4} \sum_{j=0}^{\alpha_1} a_j^2 + O(\alpha_1 \log p).$$

New task: If $\mathbb{Z}_{p^\alpha}$ is the Sylow p-subgroup of $\mathbb{Z}_n^\times$, determine the partition α (or its conjugate partition $\mathbf{a}$).

Subgroups and partitions

Lemma

For any prime $p \mid \varphi(n)$,

$$\log G_p(n) = \frac{\log p}{4} \sum_{j=0}^{\alpha_1} a_j^2 + O(\alpha_1 \log p).$$

New task: If $\mathbb{Z}_{p^\alpha}$ is the Sylow p-subgroup of $\mathbb{Z}_n^\times$, determine the partition α (or its conjugate partition $\mathbf{a}$).

How many of the factors in $\mathbb{Z}_{p^\alpha} = \mathbb{Z}_{p^{\alpha_1}} \times \cdots$ are of order at least p^j?

Subgroups and partitions

Lemma

For any prime $p \mid \varphi(n)$,

$$\log G_p(n) = \frac{\log p}{4} \sum_{j=0}^{\alpha_1} a_j^2 + O(\alpha_1 \log p).$$

New task: If $\mathbb{Z}_{p^\alpha}$ is the Sylow p-subgroup of $\mathbb{Z}_n^\times$, determine the partition α (or its conjugate partition $\mathbf{a}$).

How many of the factors in $\mathbb{Z}_{p^\alpha} = \mathbb{Z}_{p^{\alpha_1}} \times \cdots$ are of order at least p^j? We get one such factor for every prime $q \mid n$ such that $q \equiv 1 \pmod{p^j}$; this is the primary source of such factors.

Subgroups and partitions

Lemma

For any prime $p \mid \varphi(n)$,

$$\log G_p(n) = \frac{\log p}{4} \sum_{j=0}^{\alpha_1} a_j^2 + O(\alpha_1 \log p).$$

New task: If $\mathbb{Z}_{p^\alpha}$ is the Sylow p-subgroup of $\mathbb{Z}_n^\times$, determine the partition α (or its conjugate partition $\mathbf{a}$).

How many of the factors in $\mathbb{Z}_{p^\alpha} = \mathbb{Z}_{p^{\alpha_1}} \times \cdots$ are of order at least p^j? We get one such factor for every prime $q \mid n$ such that $q \equiv 1 \pmod{p^j}$; this is the primary source of such factors.

So if $\omega_{p^j}(n)$ denotes the number of primes $q \mid n$, $q \equiv 1 \pmod{p^j}$, then: $a_j = \omega_{p^j}(n)$. (This is exactly true if n is odd and squarefree, and is true up to $O(1)$ if not.) Inserting this into the above lemma...

Sketchy in the extreme

Lemma

For any prime $p \mid \varphi(n)$,

$$\log G_p(n) = \frac{\log p}{4} \sum_{j=0}^{\alpha_1} \omega_{p^j}(n)^2 + O(\alpha_1 \log p).$$

Moreover: If $p \mid \varphi(n)$ but $p^2 \nmid \varphi(n)$, then $\log G_p(n) = \log 2$.

Sketchy in the extreme

Lemma

For any prime $p \mid \varphi(n)$,

$$\log G_p(n) = \frac{\log p}{4} \sum_{j=0}^{\alpha_1} \omega_{p^j}(n)^2 + O(\alpha_1 \log p).$$

Moreover: If $p \mid \varphi(n)$ but $p^2 \nmid \varphi(n)$, then $\log G_p(n) = \log 2$.

Upon summing over all primes $p \mid \varphi(n)$:

$$\log G(n) = \sum_{p \mid \varphi(n)} \log G_p(n) \approx \log 2 \cdot \omega(\varphi(n)) + \frac{1}{4} \sum_{p^r} \omega_{p^r}(n)^2 \log p.$$

Sketchy in the extreme

Lemma

For any prime $p \mid \varphi(n)$,

$$\log G_p(n) = \frac{\log p}{4} \sum_{j=0}^{\alpha_1} \omega_{p^j}(n)^2 + O(\alpha_1 \log p).$$

Moreover: If $p \mid \varphi(n)$ but $p^2 \nmid \varphi(n)$, then $\log G_p(n) = \log 2$.

Upon summing over all primes $p \mid \varphi(n)$:

$$\log G(n) = \sum_{p \mid \varphi(n)} \log G_p(n) \approx \log 2 \cdot \omega(\varphi(n)) + \frac{1}{4} \sum_{p^r} \omega_{p^r}(n)^2 \log p.$$

Replace each of the arithmetic functions above by their known normal orders to get, for almost all n,

$$\log G(n) \approx \frac{\log 2}{2} (\log\log n)^2 + \frac{1}{4} \sum_{p^r} \left(\frac{\log\log n}{\varphi(p^r)} \right)^2 \log p = A(\log\log n)^2.$$

Future work

To handle $\log G(n)$, we approximated it by a sum of squares of additive functions.

In forthcoming work, we prove an Erdős–Kac law for arbitrary finite sums and products of additive functions satisfying standard conditions.

In other words, if $Q(x_1, \ldots, x_\ell) \in \mathbb{R}[x_1, \ldots, x_\ell]$ and $g_1, \ldots, g_\ell$ are "nice" additive functions, then $Q(g_1, \ldots, g_\ell)$ satisfies an Erdős–Kac law with a certain mean and variance.

Thanks!

Preprint available at https://arxiv.org/abs/1710.00124.