Numerical evidence for higher order Stark-type conjectures.

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Joint work with Kevin McGown and Jonathan Sands. To appear in *Mathematics of Computation*.

A common theme in the field of special values of L-functions is as follows:

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Start with an arithmetic object such as a motive, an elliptic curve, a number field, etc

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Associate to this arithmetic object an analytic object usually called a zeta or an L-function.

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Understand the value of these analytic objects at particular integers. The first non-vanishing Taylor coefficient should contain some important information related to the arithmetic object one started with.

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The study of these special values is usually done in two steps:

1. At first up to a rational number. (“Over $\mathbb{Q}$”)
2. Then understand this rational number. (“Over $\mathbb{Z}$”)

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The classical example is the class number formula for a number field K:

1. Over $\mathbb{Q}$:

$$\frac{\zeta_K^*(0)}{R_K} \in \mathbb{Q}.$$

2. Over $\mathbb{Z}$:

$$\frac{\zeta_K^*(0)}{R_K} = -\frac{h_K}{w_K}.$$

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Another famous example of such a conjecture "over $\mathbb{Z}$" is the Birch and Swinnerton-Dyer conjecture. There is also the very general Beilinson conjecture ("over $\mathbb{Q}$") starting with any motive which has been refined "over $\mathbb{Z}$" by Bloch and Kato: the Tamagawa Number Conjecture.

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All these conjectures can be studied in an equivariant way, that is incorporating the action of a Galois group as well.

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If K/k is a finite abelian extension of number fields with Galois group G and $\chi \in \widehat{G}$, Stark formulated a conjecture for the first non-vanishing Taylor coefficient

$$L^*(0, \chi),$$

which is sometimes called Stark's conjecture over $\mathbb{Q}$.

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Stark also formulated a refinement of his conjecture for imprimitive L-functions having precisely order of vanishing one at $s = 0$ (which is a conjecture "over $\mathbb{Z}$") under certain hypotheses. He proved his conjecture when the base field is $\mathbb{Q}$ and when the base field is quadratic imaginary. He provided numerical examples for the next natural case, namely for abelian extensions of real quadratic fields.

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Rubin and Popescu extended his conjecture "over $\mathbb{Z}$" to higher order of vanishing. They are known when the base field is $\mathbb{Q}$ only by works of Burns, Greither and Flach.

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The rank one abelian Stark conjecture has been numerically studied extensively by several authors, but very little numerical evidence has been provided for Rubin or Popescu's conjecture.

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We came up with a way of systematically providing numerical evidence for Rubin or Popescu's conjecture (that is the higher rank Stark conjecture "over $\mathbb{Z}$").

Numerical evidence for higher order Stark-type conjectures

We did so for 19197 examples consisting of extensions K/k, where K is a totally real abelian field that is an abelian ramified cubic extension of a real quadratic number field and whose absolute discriminant satisfies $\Delta_K \leq 10^{12}$.

The corresponding L-functions have order of vanishing two in this situation due to the two archimedean places of the bottom field k.

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The key tool is the notion of an Artin system of units. Roughly speaking they are units that generate a group of finite index on which we know how the Galois group acts. Artin gave a very concrete proof of the existence of such systems of units in a paper of his: *Über Einheiten relative galoisscher Zahlkörper*. These systems of units can be found algorithmically even though there is no canonical choice for such a system.

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We label the set of infinite places and ramified places in k:

$$S = \{v_1, v_2, \ldots, v_n\},$$

so that $|S| = n$. Following Tate, we let $Y_S(K)$ be the free abelian group on the places in S_K. We have a short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow X_S(K) \longrightarrow Y_S(K) \xrightarrow{s_K} \mathbb{Z} \longrightarrow 0, \qquad (1)$$

where the map s_K is the augmentation map and $X_S(K)$ its kernel. Recall that s_K is defined by setting $s_K(w) = 1$ for all $w \in S_K$ and extending by linearity.

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Definition

An Artin system of S_K-units $\mathcal{A}$ is a collection of S_K-units

$$\mathcal{A} = \{\varepsilon_w \,|\, w \in S_K\} \subseteq E_S(K),$$

such that the group morphism

$$f : Y_S(K) \longrightarrow E_S(K)$$

defined by $w \mapsto \varepsilon_w$ satisfies the following properties:

1. *f is G-equivariant,*
2. $\ker(f) = \mathbb{Z} \cdot \alpha$ *for some $\alpha \in Y_S(K)^G$ that satisfies $s_K(\alpha) \neq 0$.*

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Define

$$\beta_S(\mathcal{A}) = \sum_{\chi \in \widehat{G}} \frac{L^*_{K,S}(0,\chi)}{R(\chi,\mathcal{A})} e_{\overline{\chi}}.$$

Stark's conjecture over $\mathbb{Q}$ for all $\chi \in \widehat{G}$ can be phrased as follows:

$$\beta_S(\mathcal{A}) \in \mathbb{Q}[G].$$

(Analogous to $\zeta^*_K(0)/R_K \in \mathbb{Q}$.) What about those rational numbers?

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This is not yet understood, but Burns gave a conjectural bound for the denominators of

$$e \cdot \beta_S(\mathcal{A}),$$

which has something to do with the index of the group of Artin units inside the full group $E_S(K)$.

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Popescu's and Rubin's conjectures can be rephrased in terms of a property of the element

$$\beta_S(\mathcal{A}) \cdot e \cdot \varepsilon_1 \wedge \ldots \wedge \varepsilon_r \in \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} E_S(K),$$

that can be checked numerically.

The end

Thank you

Questions??

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