Western Number Theory Problems, 17 & 19 Dec 2009

Edited by Gerry Myerson

for distribution prior to 2010 (Utah) meeting

Summary of earlier meetings & problem sets with old (pre 1984) & new numbering.

1967 Berkeley	1968 Berkeley	1969 Asilomar	
1970 Tucson	1971 Asilomar	1972 Claremont	72:01–72:05
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1998 San Francisco	98:01–98:14	1999 Asilomar	99:01–99:12
2000 San Diego	000:01–000:15	2001 Asilomar	001:01–001:23
2002 San Francisco	002:01–002:24	2003 Asilomar	003:01–003:08
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2006 Ensenada	006:01–006:15	2007 Asilomar	007:01–007:15
2008 Fort Collins	008:01–008:15	2009 Asilomar	009:01–009:20

COMMENTS ON ANY PROBLEM WELCOME AT ANY TIME

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Problems proposed 17 and 19 December 2009

009:01 (Vic Dannon) Riemann, quoted on p. 838 of Hawking, ed., God Created the Integers, writes, "let us use (x) to indicate the excess of x over the next whole number, or, if x is midway between two values ... (x) indicates the average of both values $1/2$ and $-1/2$, i.e., zero." Then he lets $f(x) = \sum_{n=1}^{\infty} n^{-2}(nx)$ and writes that if $x = p/(2n)$ with p odd then

$$f(x+0) = f(x) - \frac{1}{2n^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right) \text{ and } f(x-0) = f(x) + \frac{1}{2n^2}\left(1 + \frac{1}{9} + \frac{1}{25} + \dots\right),$$

"but otherwise everywhere $f(x+0) = f(x)$, $f(x-0) = f(x)$."

How does Riemann do this?

Remark: We interpret the definition of (x) to be zero if $x = m+(1/2)$ for some integer m, otherwise $x - n(x)$, where $n(x)$ is the integer nearest x. We also interpret $f(x+0)$ (resp., $f(x-0)$) to mean $\lim_{y\to x^+} f(y)$ (resp., $\lim_{y\to x^-} f(y)$), which we will abbreviate to $f(x)^+$ (resp., $f(x)^-$).

Solution: We take it that what is asked for is a derivation of the displayed formulas. We'll do the first one, as the second follows the same lines. Interchanging limit and summation, we have

$$f(x)^+ = (x)^+ + (1/4)(2x)^+ + (1/9)(3x)^+ + \dots.$$

Note that $(y)^+ = (y) - (1/2)$ if y is half an odd integer, otherwise $(y)^+ = (y)$. Now let $x = p/(2n)$, so

$$f\left(\frac{p}{2n}\right)^+ = \left(\frac{p}{2n}\right)^+ + (1/4)\left(\frac{2p}{2n}\right)^+ + (1/9)\left(\frac{3p}{2n}\right)^+ + \dots,$$

and $(\frac{kp}{2n})^+ = (\frac{kp}{2n}) - \frac{1}{2}$ if $k = rn$ for some odd r, $(\frac{kp}{2n})$ otherwise. Thus,

$$\begin{aligned} f\left(\frac{p}{2n}\right)^+ &= \left(\frac{p}{2n}\right) + (1/4)\left(\frac{2p}{2n}\right) + (1/9)\left(\frac{3p}{2n}\right) + \dots - \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(3n)^2} + \frac{1}{(5n)^2} + \dots\right) \\ &= f\left(\frac{p}{2n}\right) - \frac{1}{2n^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \end{aligned}$$

009:02 (Russell Hendel) Let $G_n = \sum_{a=1}^{m} a_i G_{n-i}$ for some integer $m \ge 1$, with a_i real. Assume $\sum^{\infty} G_i^{-1} < \infty$. Let H_n be the nearest integer to $\left(\sum_{i=n}^{\infty} G_i^{-1}\right)^{-1}$ (rounding half-integers up). Let $T_n = H_n - \sum_{a=1}^{m} a_i H_{n-i}$.

1. Find conditions under which T_n is periodic.
2. When is there a closed form for T_n?
3. If T_n is bounded, must it be periodic?

Remark: If $G_n = c(a^n + \epsilon b^n)$ with a, b, c real, $c > 0$, $-1 < \epsilon < 1$, and $a > \max(|b|, b^2, 1)$, then T_n is bounded. A reference for related matters is

Ohtsuka and Nakamura, On the sums of reciprocals of Fibonacci numbers, Fib. Q. 46/47 (2008/2009) 153–159.

009:03 (Neville Robbins) For $1 \le k \le n$, let $\left\langle {n \atop k} \right\rangle$ be the number of cyclic equivalence classes of compositions of n into k parts. E.g., $\left\langle {6 \atop 3} \right\rangle = 4$, the four equivalence classes being those containing 411, 321, 312, and 222.

1. Find a formula for $\left\langle {n \atop k} \right\rangle$.
2. Prove that $\left\langle {n \atop n-k} \right\rangle = \left\langle {n \atop k} \right\rangle$ for $1 \le k \le n-1$.
3. Prove $\left\langle {2n \atop n} \right\rangle$ is even for all $n \ge 2$.

Remarks: It is known that if $\gcd(k,n) = 1$ then $\left\langle {n \atop k} \right\rangle = \frac{1}{n}\binom{n}{k}$, and that

$$\sum_{k=1}^{n-1} \left\langle {n \atop k} \right\rangle = -2 + \frac{1}{n}\sum_{d|n} \phi(d) 2^{n/d}$$

Solution: $\left\langle {n \atop k} \right\rangle$ counts the number of bracelets with n equally spaced beads, of which k are white, the others, black, two bracelets being considered identical if one is a rotation of the other. This solves question 2. A table of the numbers can be found at A047996 in the On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/index.html where they are referred to as "circular binomial coefficients." Many references are given, as well as the formula, $\left\langle {n \atop k} \right\rangle = (1/n)\sum_{d|\gcd(n,k)} \phi(d)\binom{n/d}{k/d}$

009:04 (Boris Kupershmidt, via Bart Goddard) Let p_n be the nth prime. It is known that $p_{n+1}(1 - p_n^{-1}) > p_n$ for $p_n > 2$. Find the largest α such that $p_{n+1}(1 - \alpha p_n^{-1}) > p_n$ for $p_n > n_0(\alpha)$.

Solution: Carl Pomerance shows that if $\alpha > 2$ then the inequality holds for all n sufficiently large, while if there are infinitely many twin primes then the inequality fails infinitely often for $\alpha = 2$.

Proof. Given any $\epsilon > 0$, we know $p_n/p_{n+1} > 1 - \epsilon$ for all n sufficiently large. Also, $p_{n+1} - p_n \ge 2$ provided $p_n > 2$. So $(p_{n+1} - p_n)p_n/p_{n+1} > 2(1 - \epsilon)$ for n sufficiently large. This is equivalent to $p_{n+1}(1 - 2(1 - \epsilon)p_n^{-1}) > p_n$ for n sufficiently large. On the other hand, if $p_{n+1} = p_n + 2$ then $p_{n+1}(1 - 2p_n^{-1}) = (p_n + 2)(1 - 2p_n^{-1}) = p_n - 4p_n^{-1} < p_n$.

009:05 (Youssef Fares) Let K be a number field. Let $E = \{ n \text{ in } \mathbf{N} : n = 2^k p_1 p_2 \dots p_r \}$ where $k \ge 0$ and $p_1, \dots, p_r$ are distinct prime numbers inert in K. Are there infinitely many n such that n and $n+1$ are both in E?

009:06 (Youssef Fares) If $f(x)$ in $\mathbf{Z}[x]$, considered as a map from $\mathbf{Z}$ to $\mathbf{Z}/p^r\mathbf{Z}$, is surjective for all primes p and all r, then the degree of f is 1. What can one conclude if f is in $\mathbf{Z}[x,y]$ and is surjective for all primes p and all r as a map from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}/p^r\mathbf{Z}$?

Remark: If $f(x,y) = x + yg(x,y)$, with g arbitrary, then $f(n,0) = n$, so f is surjective from $\mathbf{Z} \times \mathbf{Z}$ to $\mathbf{Z}$, hence to $\mathbf{Z}/p^r\mathbf{Z}$. So perhaps one cannot conclude much.

009:07 (David Terr) For positive rational α, let $g(\alpha)$ be the number of terms in the expression $\alpha = a_1^{-1} + a_2^{-1} + \dots + a_r^{-1}$ of α as a sum of unit fractions obtained by the greedy algorithm (that is, where each a_i is chosen maximal given $a_1, \dots, a_{i-1}$), and let $h(\alpha)$ be the smallest number of unit fractions summing to α. E.g., the greedy algorithm yields $9/20 = 3^{-1} + 9^{-1} + 180^{-1}$ so $g(9/20) = 3$, but $9/20 = 4^{-1} + 5^{-1}$ so $h(9/20) = 2$.

Let $d(N) = N^{-2}\#\{ (m,n) : 1 \le m < n \le N, \gcd(m,n) = 1, g(m/n) = h(m/n) \}$ (note that $\#\{ (m,n) : 1 \le m < n \le N, \gcd(m,n) = 1 \} = 3\pi^{-2}N^2(1 + o(1))$). Does $\lim_{N\to\infty} d(N)$ exist? If so, what is it? If not, what are $\limsup d(N)$ and $\liminf d(N)$?

009:08 (Carl Pomerance) Let $F^{\uparrow}(x)$ (resp., $F^{\downarrow}(x)$) be the size of the largest subset of the integers in $[1, x]$ on which the Euler phi-function is monotone non-decreasing (resp., non-increasing).

1. Is it true that $F^{\uparrow}(x) = o(x)$?
2. Is it true that $F^{\uparrow}(x) - \pi(x) \to \infty$?
3. Is it true that $F^{\downarrow}(x) = o(x)$?

Remark: It is known that there is a constant $c > 0$ such that $F^{\downarrow}(x) \geq x^c$. A conjecture of Erdős implies that this holds for every $c < 1$.

009:09 (Mike Decaro) For a given n, is there an upper bound on k, the number of consecutive primes for which

$$\left(\frac{n}{p_i}\right) = \left(\frac{n}{p_{i+1}}\right) = \ldots = \left(\frac{n}{p_{i+k-1}}\right)$$

Here $\left(\frac{n}{p}\right)$ is the Legendre symbol.

Remarks: Kjell Wooding notes the following.

1. If n is a square then clearly $\left(\frac{n}{p_i}\right) = \left(\frac{n}{p_{i+1}}\right) = \ldots = 1$ provided only that p_i exceeds the greatest prime divisor of n.

2. For any k and any p_i, we can use the Chinese Remainder Theorem to construct n such that $\left(\frac{n}{p_i}\right) = \left(\frac{n}{p_{i+1}}\right) = \ldots = \left(\frac{n}{p_{i+k-1}}\right)$.

3. Given n (not a square) and p_i, we'd expect $\left(\frac{n}{p_i}\right) = \left(\frac{n}{p_{i+1}}\right)$ about half the time, $\left(\frac{n}{p_i}\right) = \left(\frac{n}{p_{i+1}}\right) = \left(\frac{n}{p_{i+2}}\right)$ about a quarter of the time, and so on. This suggests that there is no upper bound on k.

Your editor notes that in the case $n = -1$ we are asking whether there are arbitrarily long runs of consecutive primes all belonging to the same congruence class modulo 4. Perhaps then the question is really about primes in collections of arithmetic progressions, and we could ask it this way: given a modulus m, and a proper subset S of the units modulo m, must there be arbitrarily long sequences of consecutive primes, each congruent to a unit in S?

009:10 (Gerry Myerson) Capital letters stand for finite sets of natural numbers, lower case letters for individual natural numbers. B generates a means there are subsets C and D of B such that $a = \sum(C) - \sum(D)$, where $\sum(X)$ is the sum of the elements of X. B generates A means B generates a for all a in A. Trivially, for all A, A generates A. We say A is independent if no set with fewer elements than A generates A.

1. Find a_n defined recursively as the smallest number such that $\{ a_1, a_2, \ldots, a_n \}$ is independent.

2. With a_n as above, find b_n defined recursively as the smallest r such that, for all $m \geq r$, $\{ a_1, \ldots, a_{n-1}, m \}$ is independent.

3. Find c_n defined as the smallest N such that $\{ 1, 2, \ldots, N \}$ has an independent subset with n elements.

Remarks: 1. To illustrate, $\{ 8, 9, 15 \}$ generates $\{ 1, 2, 6, 32 \}$ since $1 = 9-8$, $2 = 9+8-15$, $6 = 15 - 9$, and $32 = 15 + 9 + 8$. Thus, $\{ 1, 2, 6, 32 \}$ is not independent.

2. The a_n sequence begins $1, 2, 6, 30$. It was suggested that a_5 might be 210, but this is not the case, as $\{ 35, 36, 37, 102 \}$ generates $\{ 1, 2, 6, 30, 210 \}$. It might be the case that

$a_5 = 270$ and, generally, $a_n = \prod_{j=0}^{n-2}(2^j+1)$, but this is a hunch, not a conjecture.

3. The b_n sequence begins $1, 2, 6, 33$. We have $b_5 \geq 289$, since $\{38, 68, 75, 107\}$ generates $\{1, 2, 6, 30, 288\}$.

4. The c_n sequence begins $1, 2, 5$. Perhaps $\{6, 15, 17, 18\}$ is independent, and perhaps $c_4 = 18$.

009:11 (M. Tip Phaovibul) Let ϕ be the Euler phi-function, let $S_n = \sum_{i=1}^{n} \phi(n)$, let p be an odd prime, and let $A_a = \{ n : S_n \equiv a \pmod p \}$.

1. Does A_a have positive density in $\mathbf{N}$?

2. Is S_n uniformly distributed (modulo p)? That is, do we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{ n \leq N : S_n \equiv a \pmod p \} = \frac{1}{p}$$

for all a?

009:12 (Roger Baker) Let $\mathcal{S}$ be a sequence $a_1, a_2, \ldots$ of positive integers, let I be a subinterval of $[0,1]$, and let $E_{\mathcal{S}}(I) = \{ x \text{ in } \mathbf{R} : \{a_n x\} \text{ is not in } I, n = 1, 2, \ldots \}$, where $\{y\}$ is the fractional part of y.

1. Show that if $a_n = O(n^p)$ for any $p > 1$ then the Hausdorf dimension of $E_{\mathcal{S}}(I)$ is zero.

2. Construct a sequence with $a_n = O(n^p)$ for some $p > 1$ such that $E_{\mathcal{S}}(I)$ is uncountable for some I.

009:13 (Youssef Fares) Let p be a prime and let F_m and F_n be Fibonacci numbers. Write $\nu_p(r)$ for the number s such that p^s divides r but p^{s+1} doesn't. What is $\nu_p(F_n - F_m)$?

009:14 (Bart Goddard) For k in $\mathbf{N}$, what are the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \binom{2n+1}{2k} \frac{y^{2n-1}}{(2n+1)!} \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} \binom{2n+1}{2k+1} \frac{y^{2n-2}}{(2n+1)!}$$

Remarks: 1. For $k = 1$, the first series is $\sin y$, and for $k = 0$, the second series is $\cos y$.

2. It was suggested that it might be possible to express the sums as hypergeometric functions.

009:15 (Christina Holdiness) Let p_i be the ith prime. Is $p_1 p_2 \ldots p_n - p_{n+1}$ prime?

Solution: Jianqiang Zhao found the first counterexample: $2\times3\times5\times7\times11\times13\times17-19 = 41 \times 12451$. Perhaps one could ask whether infnitely many of these numbers are prime.

009:16 (Nathan Rowe) Is it true that for every natural number n and for every m in $\mathbf{Z}/n\mathbf{Z}$ there is a polynomial f with coefficients in $\mathbf{Z}/n\mathbf{Z}$ such that $f(m) = 1$ and $f(x) = 0$ for $x \neq m$?

Solution: If n is composite then there exists m in $\mathbf{Z}/n\mathbf{Z}$ for which there is no such polynomial. For let $n = rs$ with $r > 1$ and $s > 1$. Then $f(r) \equiv f(0) \pmod r$.
If $f(r) \equiv 1 \pmod n$, then $f(r) \equiv 1 \pmod r$, so $f(0) \equiv 1 \pmod r$, so $f(0) \not\equiv 0 \pmod n$.

On the other hand if n is prime then $\prod_{a\neq m} \frac{x-a}{m-a}$ is such a polynomial.

009:17 (Jianqiang Zhao) Let B_n be the nth Bernoulli number. Is it true that for all prime $p \geq 11$,

$$\sum_{1\leq i<j<k<\ell\leq p-1} \frac{1}{i^3 j k^3 \ell} \equiv -\frac{p}{72} B_{p-9} \pmod{p^2}$$

009:18 (Jean-Marie De Koninck and Nicolas Doyon) Let $P(n)$ be the largest prime dividing n, and let $\delta(n)$ be the distance from n to the nearest integer m with $P(m) \leq P(n)$.

1. Prove that for all $k \geq 1$ the expected proportion of integers n such that $\delta(n) = k$ is $2/(4k^2 - 1)$.

2. Given k, let $n = n_k$ be the smallest positive integer such that $\delta(m) = 1$ for all m, $n \leq m \leq n + k - 1$. Is it true that $n_k \leq n!$ for all $k \neq 4$?

3. Let $\Delta(n) = \sum_{d|n} \delta(d)$. Given k, let $n = n_k$ be the smallest n such that $\Delta(n) = \Delta(n+1) = \ldots = \Delta(n+k-1)$. Does n_k exist for all $k \geq 2$?

Remarks: 1. To illustrate, here is a table to show that $\delta(100) = 4$.

n	96	97	98	99	100	101	102	103
$P(n)$	3	97	7	11	5	101	17	103

2. The first part of the question is implied by the following hypothesis: let k be at least 2, and let $a_1, a_2, \ldots, a_k$ be any permutation of the numbers $0, 1, \ldots, k-1$. Then we have $\text{Prob}\big(P(n + a_1) < P(n + a_2) < \ldots < P(n + a_k)\big) = 1/k!$.

3. Here is a small table of values of n_k for the second question.

k	1	2	3	4	5	6	7	8	9	10	11	12
n_k	1	1	1	91	91	169	2737	26536	67311	535591	3021151	26817437

k	13	14	15
n_k	74877777	657240658	785211337

4. For the third problem we have $n_2 = 14$ ($\Delta(14) = \Delta(15) = 4$), $n_3 = 33$ ($\Delta(33) = \Delta(34) = \Delta(35) = 4$), $n_4 = 2189815$ ($\Delta(n_4 + i) = 12$ for $i = 0, 1, 2, 3$), $n_5 = 7201674$ ($\Delta(n_5 + i) = 14$ for $i = 0, 1, 2, 3, 4$), and n_6, if it exists, exceeds 1,500,000,000.

009:19 (Dave Rusin) Hayes (anticipated, at least in part, by Bredihin) proved that if $f(x)$ is of degree $n \geq 1$ in $\mathbf{Z}[x]$ then $f = g + h$ for some irreducible polynomials g and h, each of degree n. Saidak, attributing the result to Hayes, proved that if $f(x)$ is monic with degree at least 1 then $f = g + h$ for some irreducible monic polynomials g and h (but if the degree of f is 1 then this seems to require us to accept the constant polynomial 1 as irreducible). Under what conditions on f can we insist that g and h have non-negative coefficients? For example, is it true if f is monic with non-negative coefficients at least three of which, including the constant term, exceed 1?

009:20 (Pante Stanica) For k and t natural numbers let S_t be the set of pairs (a, b), $0 \leq a, b \leq 2^k - 2$, such that $a + b \equiv t \pmod{2^k - 1}$ and $s_2(a) + s_2(b) < k$, where $s_2(n)$ is the number of ones in the binary representation of n. Show that $\#(S_t) < 2^{k-1}$.

Remark: This has been verified for $t \leq 19$ and also for all t of various special forms.