# Periods Relations for Riemann Surfaces with Many Automorphisms

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$$\Delta = \langle \delta_p, \delta_q, \delta_r \,|\, \delta_p^p = \delta_q^q = \delta_r^r = \delta_p \delta_q \delta_r = 1 \rangle.$$

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$\Delta$ acts on the complex upper half-plane $\mathbb{H} = \{x+iy \in \mathbb{C} \mid y > 0\}$ by linear fractional transformations,

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If $G$ is any finite index subgroup of $\Delta$, then the orbit space $X(G) = G \backslash \mathbb{H}$ for this action carries the structure of a compact Riemann surface, called the **automorphic curve** associated to $G$.

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Identifying $\Omega^1(X(N))$ with the space $M_2(N)$ of weight two automorphic forms for $N$, this action is given by $f \mapsto f|_2\gamma$ where

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$$f|_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right)(c\tau + d)^{-2}.$$

This yields a $g$-dimensional complex representation $\rho_N : \Delta \to GL(M_2(N))$, called the **canonical representation** of the cover $X(N) \to X(\Delta)$.

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These numbers span a full rank lattice $\Lambda$ in $\mathbb{C}^g$ (i.e. a free $\mathbb{Z}$-module of rank $2g$), and $\mathbb{C}^g/\Lambda$ is an **abelian variety** called the **Jacobian** of $X(N)$.

# Period relations for $X(N)$

Let $V_N = span_{\overline{\mathbb{Q}}}\{\omega_{jk} \,|\, 1 \leq j \leq 2g,\ 1 \leq k \leq g\}$ be the $\overline{\mathbb{Q}}$-span of the periods of $X(N)$, viewed as a subspace of $\mathbb{C}$.

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Although not particularly strong, it already implies nontrivial arithmetic results.

# Example: Elliptic curves with many automorphisms

Suppose the genus of $X(N)$ is 1, so that $X(N)$ defines an elliptic curve $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ denotes the period lattice of $X(N)$.

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By Wolfart's bound we have $\dim_{\overline{\mathbb{Q}}} V_N = g = 1$, so $\frac{\omega_1}{\omega_2} \in \overline{\mathbb{Q}}$.

In other words, $X(N)$ is an elliptic curve with **complex multiplication** and, as is well-known the above period ratio lies in a quadratic imaginary extension of $\mathbb{Q}$.

# A new bound on $\dim_{\overline{\mathbb{Q}}} V_N$

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By studying the cohomology group $H^1(\Delta, \rho_N)$ for the canonical representation of the covering $X(N) \to X(\Delta)$, one may obtain the following result:

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**Theorem**

*For $j = p, q, r$ let $d_j$ denote the dimension of the eigenspace of $\rho_N(\delta_j)$ associated to the eigenvalue 1. Then*

$$\dim_{\overline{\mathbb{Q}}} V_N \leq g - d_p - d_q - d_r.$$

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Note that, on average, this gives a bound of

$$g - \frac{g}{p} - \frac{g}{q} - \frac{g}{r} = g\left[1 - \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)\right] \ll g.$$

# A genus two example

The largest possible automorphism group for a genus two algebraic curve is GL(2,3), and this is realized by a normal subgroup $N \triangleleft \Delta = \Delta(2,3,8)$ of index 48. $X(N)$ is called a *Bolza surface*.

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The canonical representation $\rho_N$ for the covering $X(N) \to X(\Delta)$ is irreducible, and this may be combined with a theorem of Shiga and Wolfart to show that the Jacobian of $X(N)$ factors as $E^2$ for an elliptic curve $E$ with complex multiplication.

# Don't call it a conjecture

The proof of the above bounding theorem shows that when $X(N)$ has many automorphisms then

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Might this be true for all curves with many automorphisms?

If so, then this would provide a powerful method for determining when Jacobians for curves with many automorphisms have complex multiplication.

Thanks very much!