The Prime Number Theorem for Juggling Patterns

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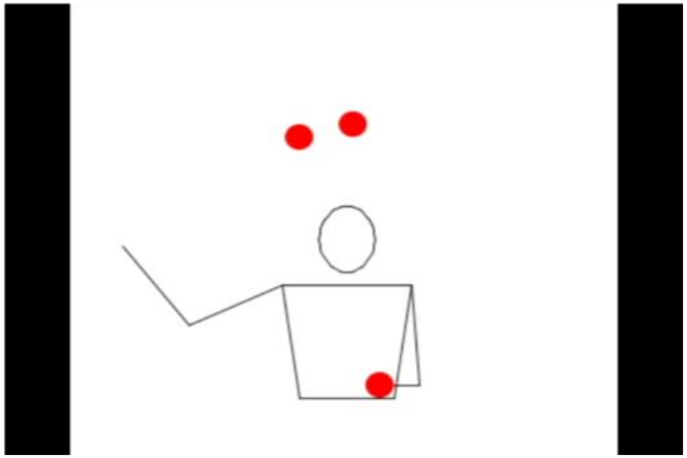
- Measure *height* of throw according to number of "beats" until it comes back down (usually, "beats" = "thuds")



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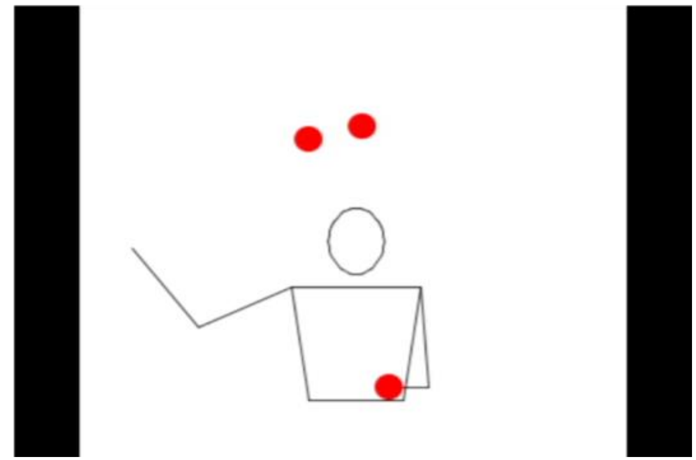

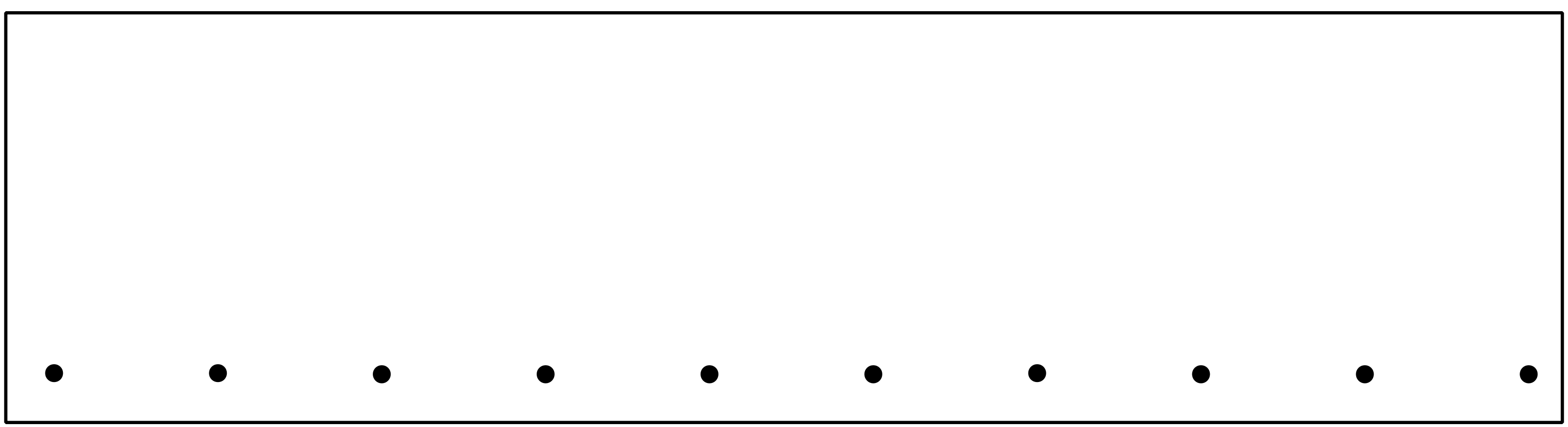

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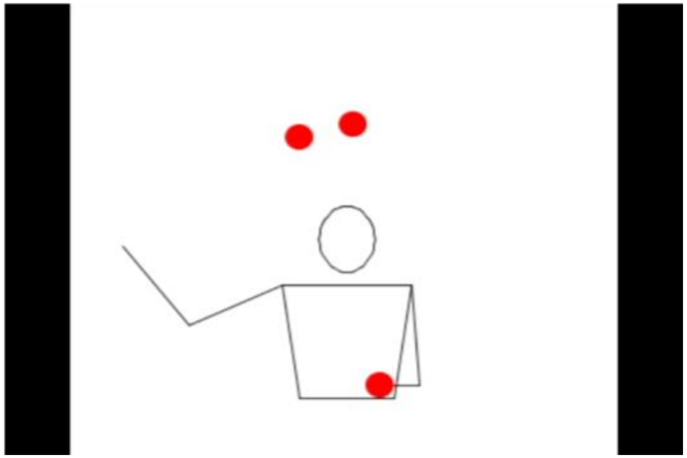

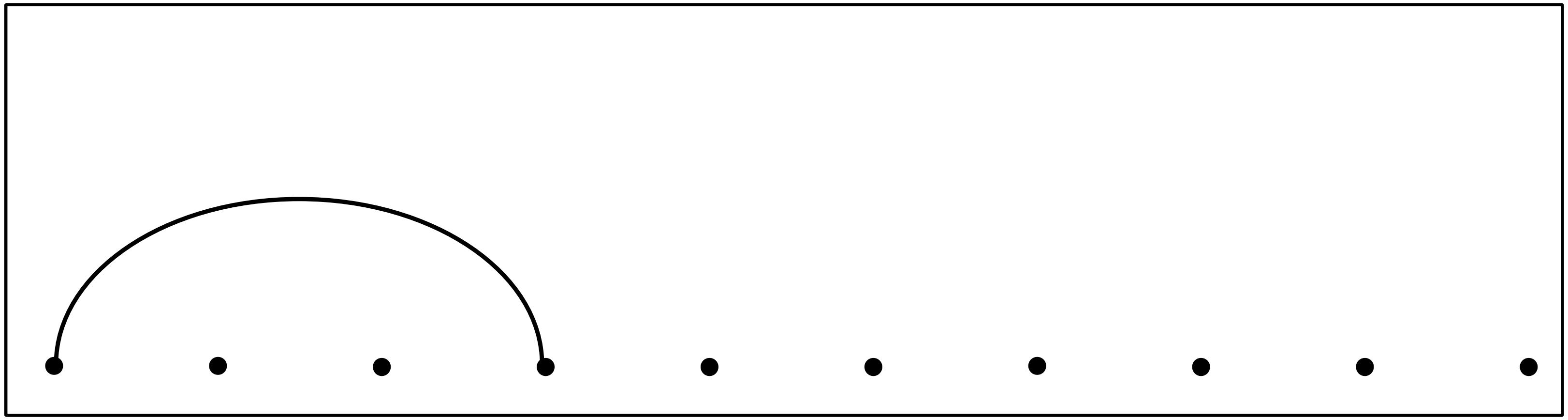

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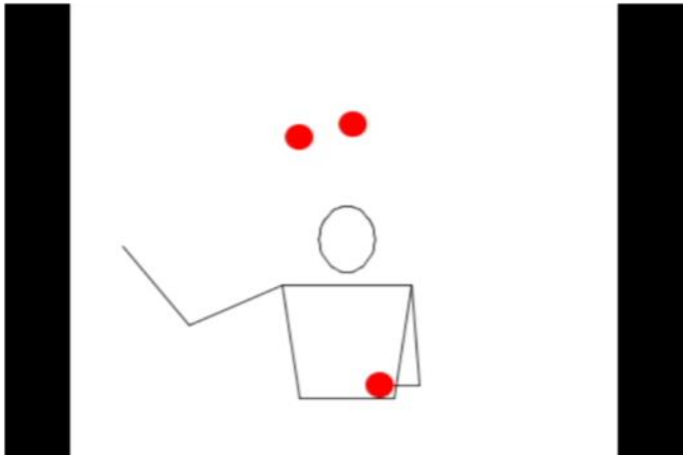

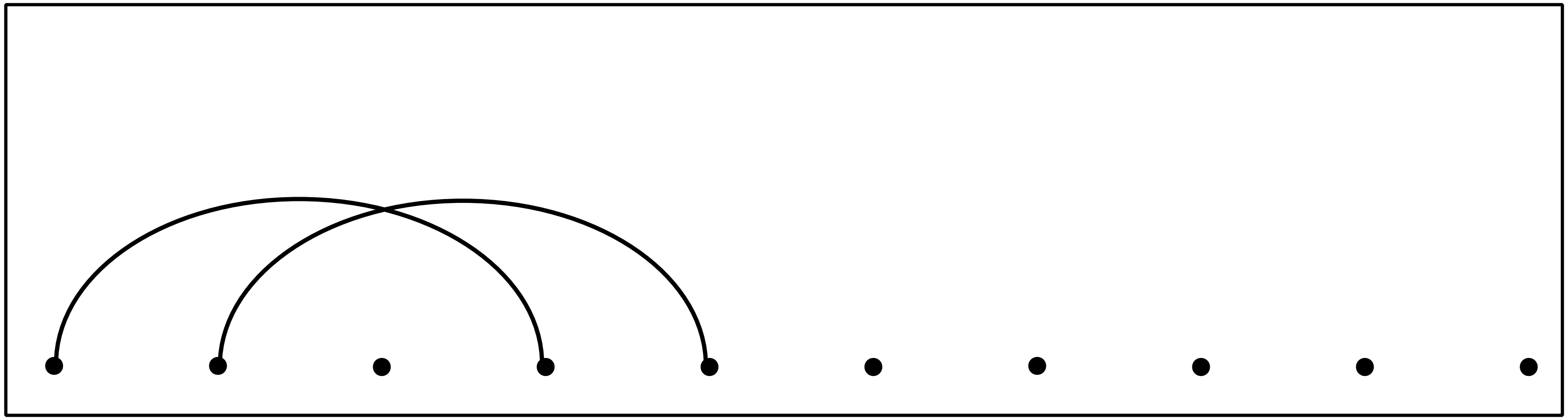

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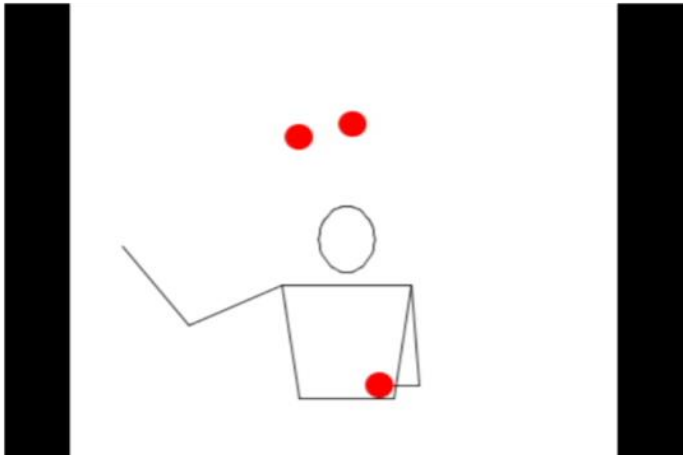

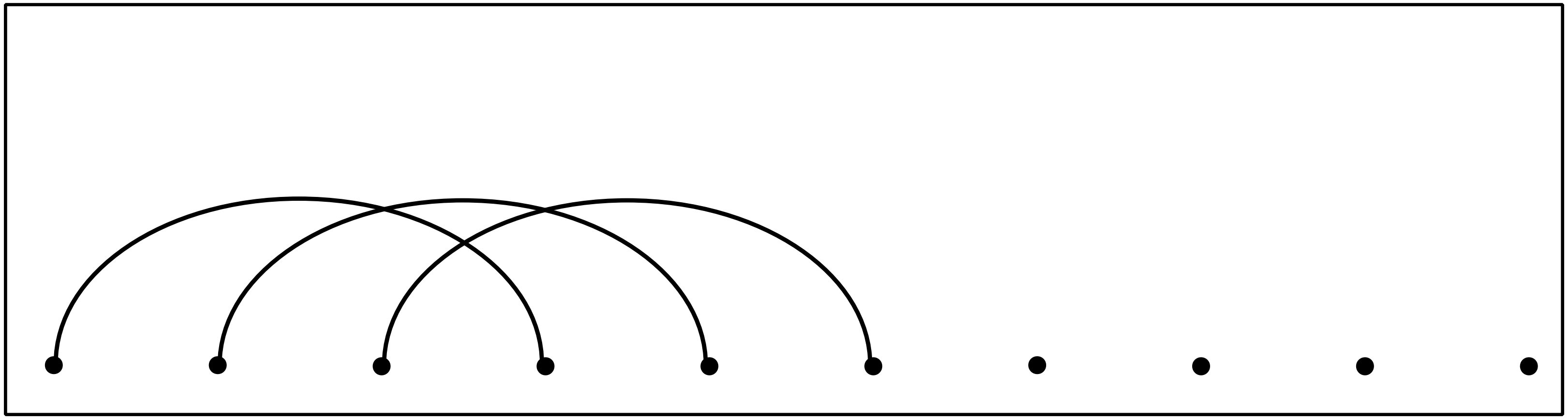

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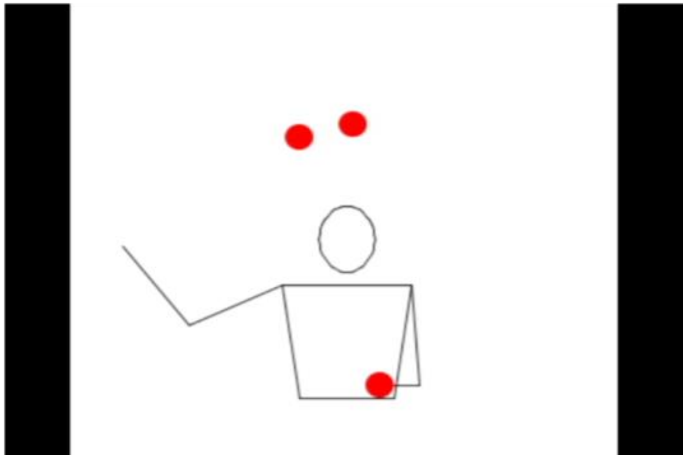

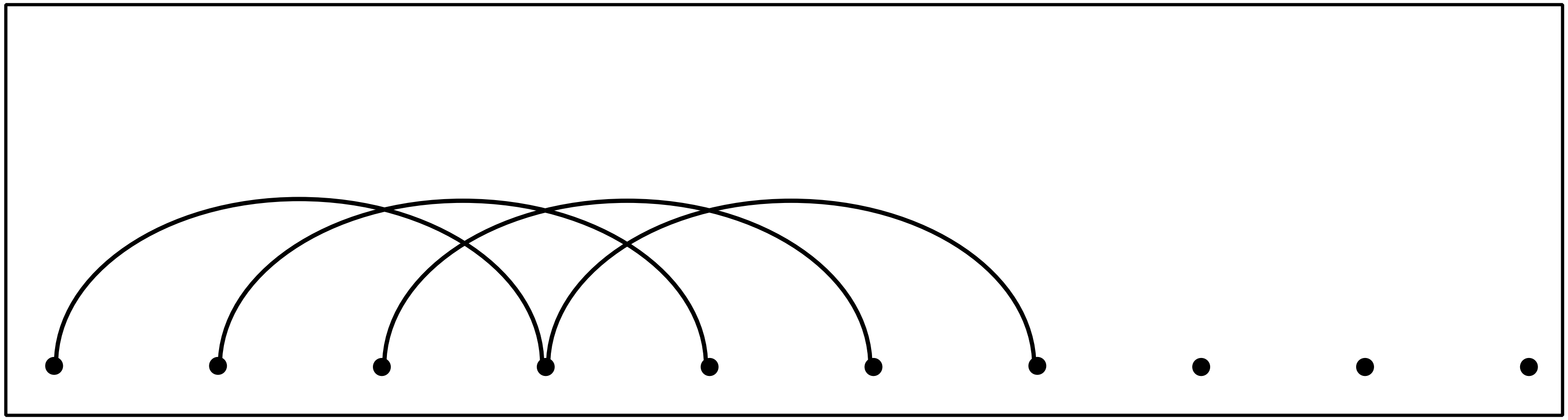

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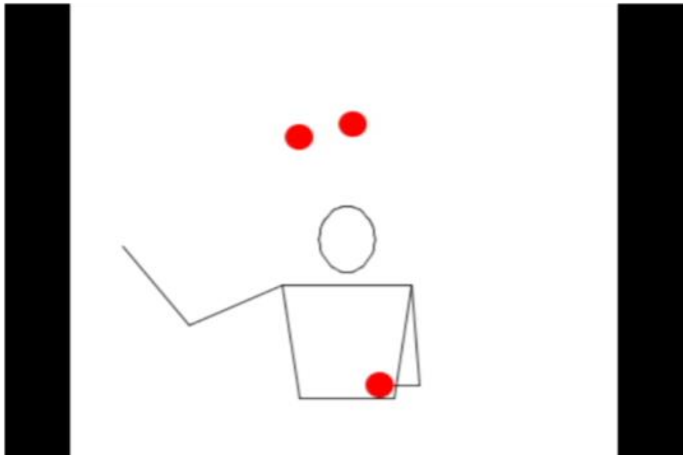

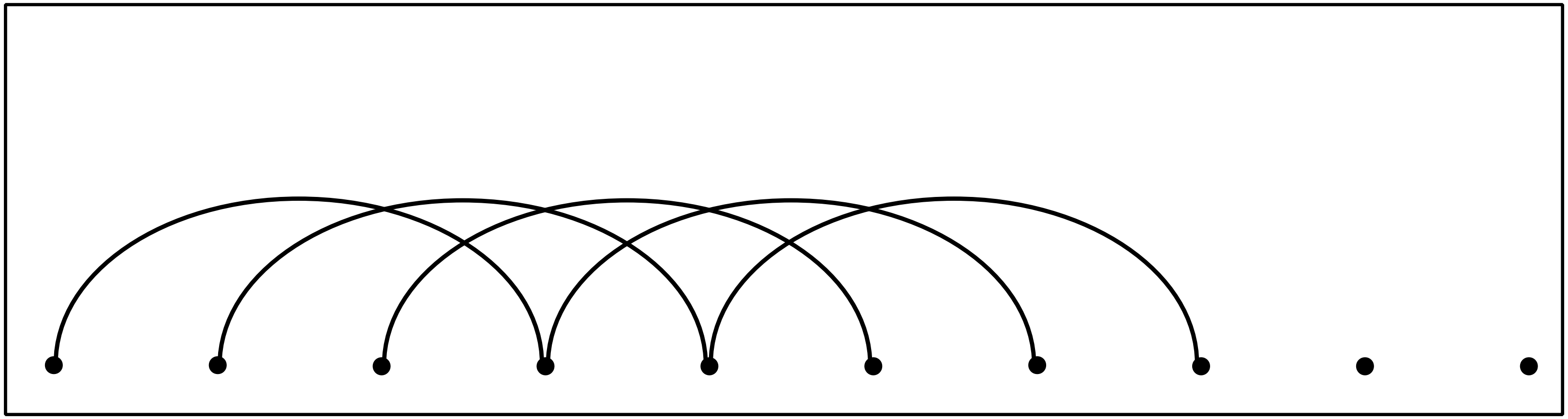

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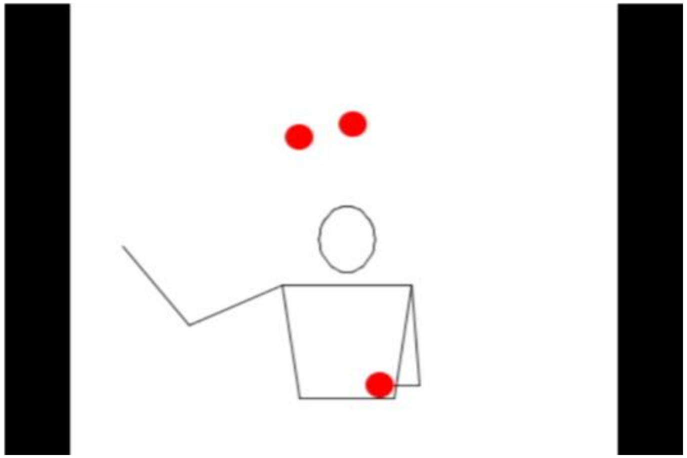

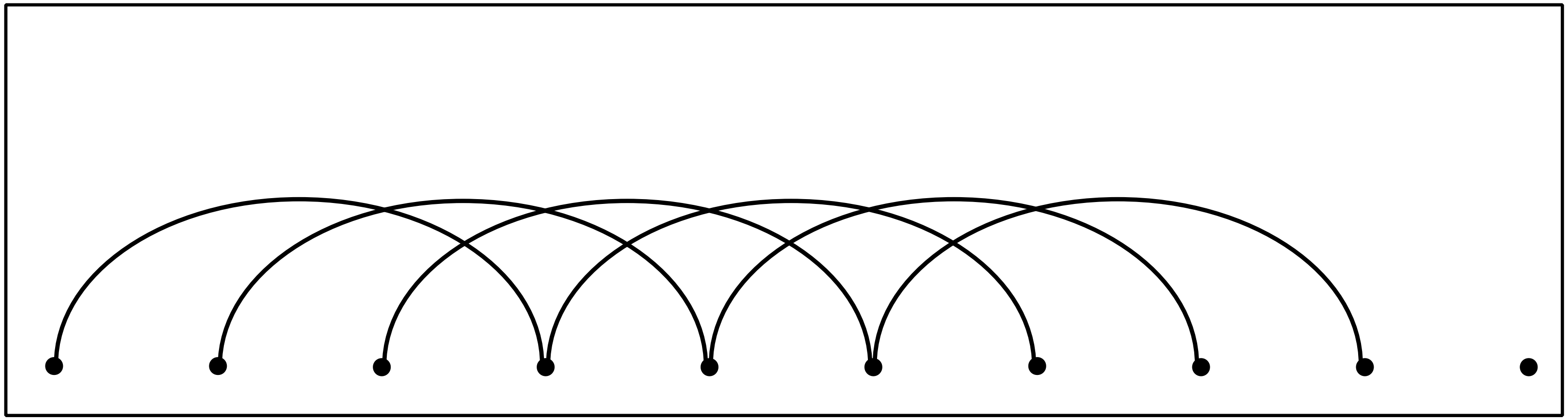

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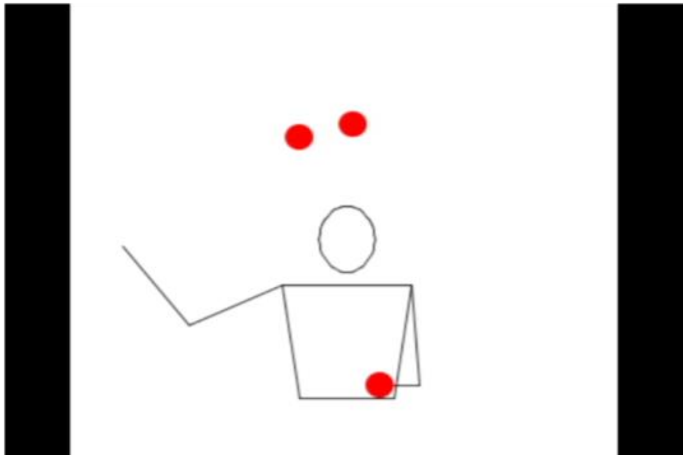

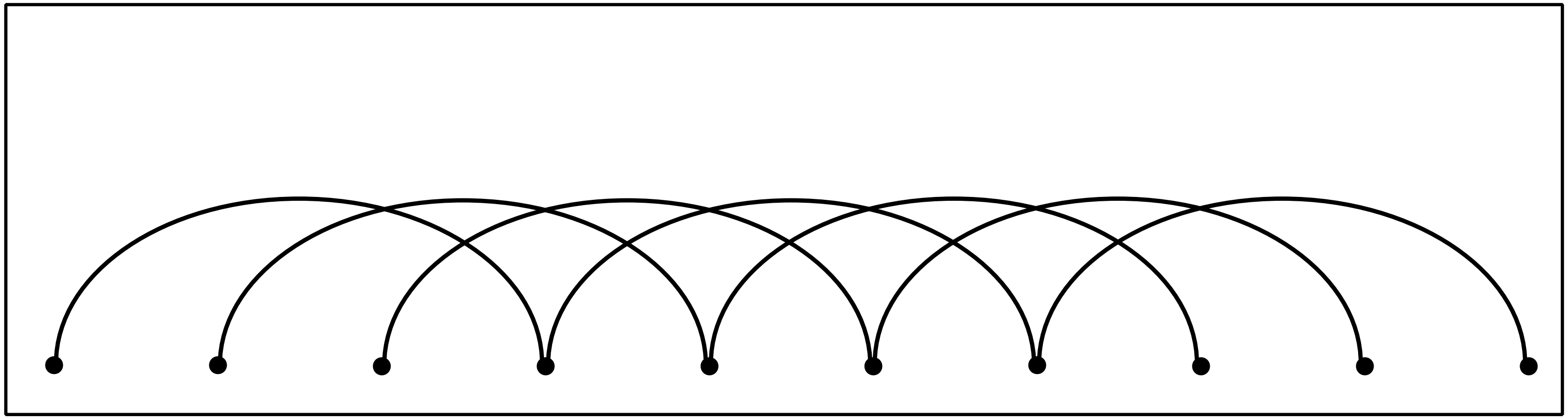

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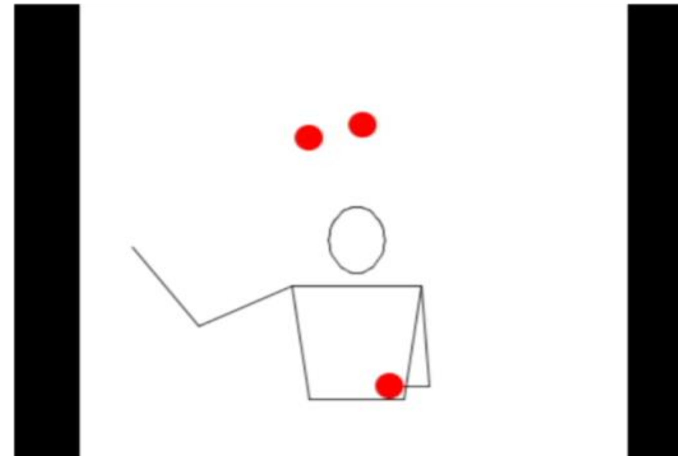
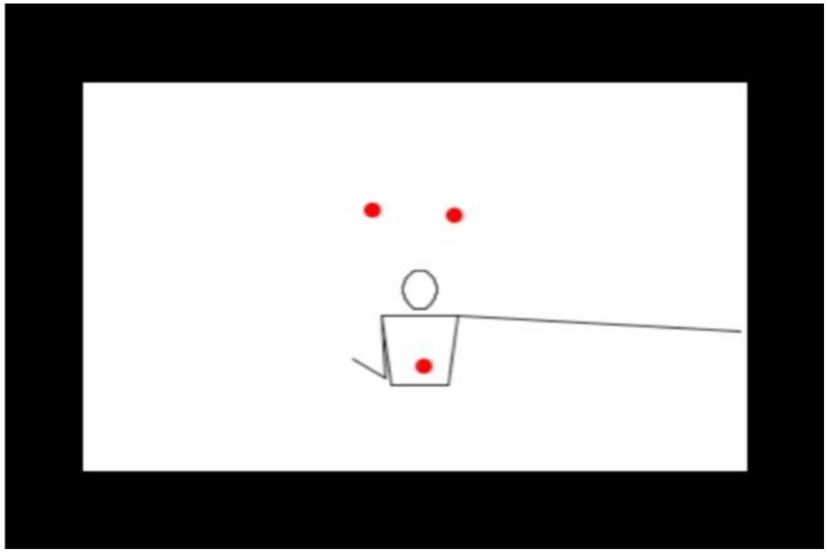
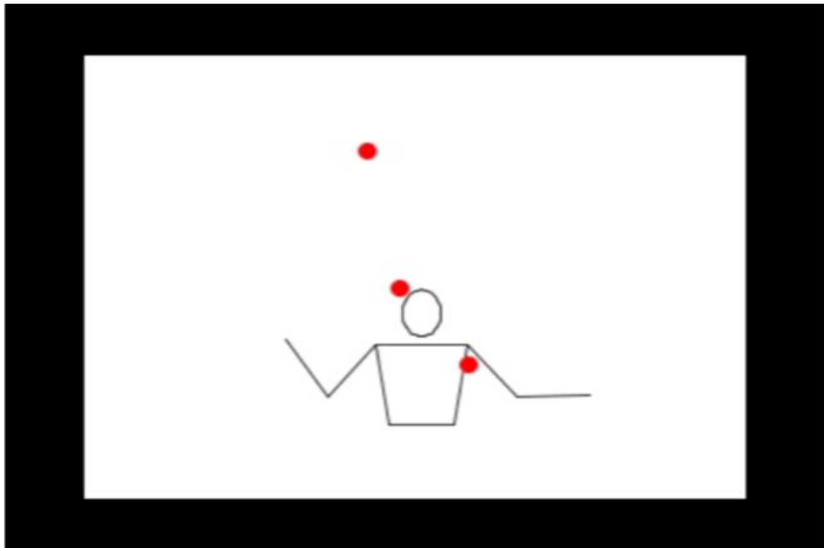
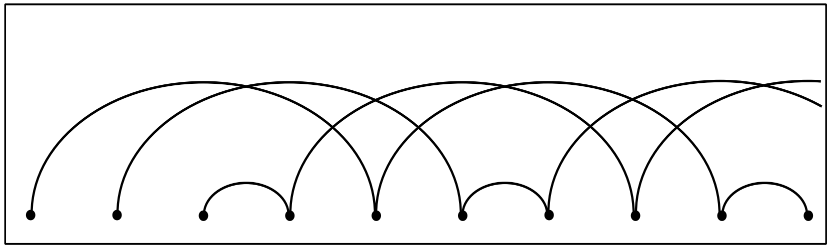


Repeated throws of height 3 → *Siteswap* notation is (3)

More Examples — (441) and (531)







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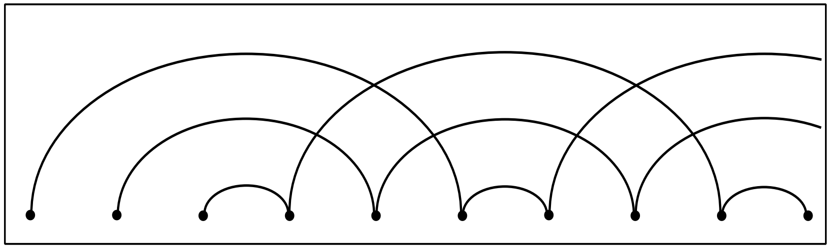

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Juggling Sequences and Facts

Definition: A *juggling sequence* can be specified by a list of non-negative *heights* $(t_1, t_2, \dots, t_n)$, with the restriction that $i + t_i \not\equiv j + t_j$ mod *n* for any $1 \leq i, j \leq n$.

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Number Theory Facts:

- We can multiply "compatible" juggling patterns by concatenation.
- *Ground state* patterns are those compatible with (*b*).
- *Primitive* juggling patterns are those which cannot be factored into smaller patterns.

Juggling Number Theory

An Example: For $b = 3$, let $j = (5340342623 1)$.

- This factors as (5340)(3426231), so we say (5340) *divides* (53403426231).
- Unique factorization into primitive sequences:

$$j = (5340)(3)(42)(6231).$$

- Primitive sequences: (3), (42), (441), (531), (522), …

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The Takeaway: Ground state juggling sequences are a set in which we can multiply (non-commutatively) and factor (uniquely), with a subset that serves as an analogue to prime numbers.

Counting Juggling Patterns

Q: How many b-ball juggling patterns are there of length n?

A: Fortunately, this is known [Chung & Graham, 2008]:

$$J_b(n) = \begin{cases} n! & \text{if } n < b \\ b! \cdot (b+1)^{n-b} & \text{if } n \geq b. \end{cases}$$

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This allows us to determine a closed form for the generating function $f_b(x) = \sum J_b(k)\, x^k$:

$$\begin{aligned} f_b(x) &= \sum_{k<b} k!\, x^k + \sum_{k \geq b} b!\, (b+1)^{k-b} x^k \\ &= r_b(x) + \frac{b!\, x^b}{1-(b+1)x}. \end{aligned}$$

Counting Primitive Juggling Sequences

The number $P_b(n)$ of primitive juggling sequences can be computed via a *b*-fold recurrence, which allows us to express the generating function $g_b(x)$ for primitive *b*-ball juggling sequences in terms of $f_b(x)$:

$$g_b(x) = \frac{f_b(x) - 1}{f_b(x)}$$

$$= \frac{r_b(x)(1-(b+1)x) + b!\,x^b - (1-(b+1)x)}{r_b(x)(1-(b+1)x) + b!\,x^b}$$

$$= \frac{s_b(x) - (1-(b+1)x)}{s_b(x)}$$

Counting Asymptotically

Problem: Computing $P_b(n)$ with a b-fold recurrence or a generating function is *hard*.

Solution: View $g_b(x)$ as an analytic object, $g_b(z)$. Then the asymptotic growth of $P_b(n)$ depends on the pole of $g_b(z)$ with smallest modulus.

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Theorem. [Flajolet & Sedgwick, 2009] Let $g_b(z) = \sum c_k z^k$ be a generating function viewed analytically, with a single minimal pole *d* (by modulus). Then,

$$c_n \sim \operatorname*{Res}_{z=d} g_b(z) \cdot d^{1-n}$$

Main Result

Theorem [τ, 2017]. Suppose $b \geq 4$, and let d be the smallest root (by modulus) of $s_b(z)$. Then,

$$P_b(n) \sim \frac{b+1-d^{-1}}{|s_b{}'(d)|} \cdot d^{2-n}$$

where

$$b+1-6.04 \cdot \frac{b^{3/2}}{e^b} \leq d^{-1} < b+1.$$

Conjectured, but not quite proven:

$$d^{-1} < b+1-c \cdot \frac{b^{1/2}}{e^b}.$$

Idea of the Proof

Recall that $s_b(z) = b!\,x^b + (1-(b+1)z)r_b(z)$.

- A half-truth: The roots of $r_b(z) = \sum_{k=0}^{b-1} k!\,z^k$ are approximately equal to the roots of $\sum_{k=0}^{b-1} \mu^k\,z^k$, where $\mu = \sqrt[b-1]{(b-1)!}$, so they have modulus $\approx \mu^{-1}$.

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- The other root d of $s_b(z)$ is approximated by $(1-(b+1)z) = 0$, so that $d \approx (b+1)^{-1}$.
- Since $(b+1)^{-1} \ll \mu$, the smallest root of $s_b(z)$ is d, so it governs the asymptotic growth of $P_b(n)$.

Computing the Residue

$$\operatorname*{Res}_{z=d} g_b(z) = \lim_{z\to d}(z-d)\cdot\frac{s_b(z)-(1-(b+1)z)}{s_b(z)}$$

$$= \lim_{z\to d}\frac{s_b(z)-(1-(b+1)z)}{(s_b(z)-s_b(d))/(z-d)}$$

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The asymptotic result follows:

$$P_b(n) \sim \frac{(b+1)-d^{-1}}{|s_b{}'(d)|}\cdot d^{2-n}$$

Open Questions

- Can we get better bounds for the size of *d*?
- Given a juggling sequence *j* with length *n*, how many sequences of length ≤ *n* are "relatively prime" to *j*?
- What happens when you allow for a ball to be added or dropped (i.e., what if *b* can change)?
- There are *prime* juggling sequences (defined as cycles on a *state graph*). Can those be counted in a similar way?

References

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Thank you!