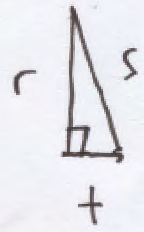
Integer Points on $E_m : y^2 = X^3 - mX$

- $m = n^2$, E_{n^2} is a congruent number curve



r, s, t rational iff $\mathrm{rank}(E_{n^2}) > 0$
AREA $= n$

- Determine $E_m(\mathbb{Z})$
  - W : $m = p^a$
  - Bennett : $m = 2^a p^b$
  - THIS TALK : $m = p^a q^b$ p, q odd primes

CHEEKY SOLUTIONS on $E_m(\mathbb{Z})$

If $p \mid m$ and $(x,y) \in E_m(\mathbb{Z})$
then $(p^2 x, p^3 y) \in E_{p^4 m}(\mathbb{Z})$.

<u>Def'n</u> $(x,y) \in E_m(\mathbb{Z})$ is <u>primitive</u> if
$p^2 \nmid x$ for any $p \mid m$.

<u>Problem</u> Determine the set of $(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$
for which there exist curves

$$E_{p^a q^b} \qquad (p, q \text{ distinct odd primes})$$

with primitive solutions.

$(a,b) = (1,1)$

$$y^2 = x^3 - pqx$$

$$= x(x^2 - pq) = du^2 \cdot dv^2$$

Let $x = du^2$, d square-free

So $d \in \{\pm 1, \pm p, \pm q, \pm pq\}$

$d=1$

$$u^4 - pq = v^2$$

$$\hookrightarrow \begin{aligned} u^2 \pm v &= p \ , \ pq \\ u^2 \mp v &= q \ , \ 1 \end{aligned}$$

$$\hookrightarrow \quad 2u^2 = p+q \quad \text{or} \quad pq+1 \qquad \circledast$$

Point: Conjecturally, there infinitely many (p,q) which satisfy $\circledast$.

$d = -1$

$$u^4 + v^2 = pq$$

- A modification to Friedlander & Iwaniec's Theorem would give infinitely many

$$E_{pq}$$

with a primitive integer point.

(TENTATIVE) THEOREM

There are infinitely many odd prime pairs p, q s.t.

$$E_{pq}$$

has a primitive integer point.

$(a,b)=(1,t)$ $\quad t>1.$

$$y^2 = x^3 - pq^t x$$
$$= x(x^2 - pq^t)$$

$x = du^2$, $d \in \{\pm 1, \pm p, \pm q, \pm pq\}$.

Case $d=1$

Same reduction $\longrightarrow$ $2u^2 = p + q^t$

- Select a prime q
- Put $u = \lceil \sqrt{q^t/2} \rceil$
- Test $2(u+i)^2 - q^t$ for primality.

Gives E_{pq^t} with a primitive integer point.

- SIMILARLY for p^2q^t, $t \geq 2$.

Conjecture

Let $\mu = 1$ or 2 and $t \geq 1$ be an integer.
There are infinitely many odd prime pairs (p, q) for which

$$E_{p^{\mu} q^{t}}$$

has a primitive integer point.

$(a, b) = (3, 3)$

As before, with $d = 1$ gives

$$2u^2 = p^3 + q^3$$

$$f_1 = 4c^4 + 12c^2d^2 - 3d^4$$
$$f_2 = -4c^4 + 12c^2d^2 + 3d^4$$
$$f_3 = 24c^5d + 18cd^5$$

Then $2f_3^2 = f_1^3 + f_2^3$

AND Moreover: f_1 and f_2 are irreducible!

Conjecture: Infinitely many odd prime pairs (p, q) exist such that $E_{p^3q^3}$ contains a prim. int. pt.

$(a,b) = (3,4), (3,5)$

$d=1 \rightarrow \quad 2u^2 = p^3 + q^4$

$$2u^2 = p^3 + q^5$$

Corresponding curves can ALSO be parametrized BUT TBD whether irreducibility holds.

OPEN PROBLEM

Are there infinitely odd prime pairs (p,q) for which

$$E_{p^3q^4} \quad \text{resp.} \quad E_{p^3q^5}$$

contains a primitive integer point?

$(a,b)=(3,6)$ Rank $=1$

$d=1 \to 2u^2 = p^3+q^6 \to y^2=X^3+8$

Need $X=\frac{2p}{q^2}$ NONE found!!

$d=-1 \to u^4+v^2=p^3q^6$

$x^4+y^2=Z^3$ parametrizable

need $Z=pq^2$

OPEN PROBLEM are there any (p,q) such that $E_{p^3q^6}$ has a prim. int. pt.?

ABC IMPLIES (VIA Darmon & Granville)

$$L x^r + M y^s = N z^t \qquad (*)$$

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} > 1$$ Spherical case $g=0$ (parametrizable)

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$$ Euclidean Case $g=1$

Eg $(a,b) = (3,6)$
$(a,b) = (4,4)$

$$(**) \quad \frac{1}{r} + \frac{1}{s} + \frac{1}{t} < 1$$ Hyperbolic Case $g>1$

ABC CONJECTURE $\Rightarrow$ for any (L, M, N) there are only finitely many (x, y, z, r, s, t) satisfying $(*)$ and $(**)$.

$\underline{(a,b) = (3,7)}$

$\boxed{\frac{1}{2}+\frac{1}{3}+\frac{1}{7} < 1}$

$\underline{d=1} \longrightarrow 2z^2 = p^3 + q^7$

has a sol'n : $(p,q,z) = (5,3,34)$.

$$2 \cdot 34^2 = 5^3 + 3^7$$

"Good" ABC EXAMPLE

$\underline{\text{Note}}$ $\dfrac{\log(2 \cdot 34^2)}{\log(2\cdot 3 \cdot 5 \cdot 17)} = 1.242\ldots$

BOLD
Conjecture

AND $(a,b) \neq (3,6)$

If $\frac{1}{a} + \frac{1}{b} \le \frac{1}{2}$

and (x,y) is a prim. int. point on $E_{p^a q^b}$,

then $p^a = 5^3$, $q^b = 3^7$, $x = 2 \cdot 34^2$.

HOW ABOUT A THEOREM? (DIAGONAL CASE)

If $a = b$ and $a, b \geq 4$ then

$$E_{p^a q^b}$$

does NOT HAVE any primitive int. points.

Direct APPLICATION of

- BENNETT & SKINNER $\quad 2z^2 = x^n + y^n \quad (n \geq 4)$
- BENNETT, ELLENBERG, NG $\quad x^2 + y^4 = z^n \quad (n \geq 4)$