The k-fold Divisor Function in Arithmetic Progression to Large Moduli

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Background–Where does the sum come from?

- **Gauss** (∼1792) conjectured the **Prime Number Theorem**:

$$\textbf{Gauss}\ (1792): \quad \sum_{p \le X} 1 \ \sim \ \frac{X}{\log X}.$$

$$\sum_{p \le X} 1 \to \sum_{p \le X} \log p \to \sum_{p^\alpha \le X} \log p = \sum_{n \le X} \Lambda(n) \to \sum_{\substack{n \le X \\ n \equiv a \pmod d}} \Lambda(n) \to \sum_{\substack{n \le X \\ n \equiv a \pmod d}} f(n).$$

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- For us, we consider the **k-fold divisor function**

$$f(n) = \tau_k(n) = \sum_{\substack{d_1 d_2 \cdots d_k = n \\ d_i > 0}} 1, \quad k \ge 1.$$

Thus, $\tau_k(n)$ is the coefficient of n^{-s} in the Dirichlet series

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- **Ultimate goal:** For each $k \ge 1$, $d \ge 1$, $(a, d) = 1$, estimate the sum

$$\sum_{\substack{n\le X \\ n\equiv a (\text{mod } d)}} \tau_k(n)$$

for d as large as possible with an error as small as possible.

Why $\tau_k(n)$?

- It turns out that $\tau_k(n)$ is closely related to **primes**, e.g., Linnik identity (1963), Vaughan identity (1977), Heath-Brown identity (1982), etc... Heuristically,

$$\frac{1}{\zeta(s)} = \frac{1}{1-(1-\zeta(s))} = \sum_{n=1}^{\infty}(1-\zeta(s))^n = \sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\zeta(s)^k,$$

so a hard problem with primes on the left side is converted into infinitely many, though more accessible, problems about τ_k on the right side.

[1] J. B. Friedlander and H. Iwaniec, Ann. of Math. (2) 121 (1985), **no. 2**, 319-350.
[2] D. Goldston, J. Pintz, and C. Yıldırım, Ann. of Math. 170 (2009), **no. 2**, 819-862.
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- Concretely,

Friedlander & Iwaniec [1] (1985)
$\tau_3(n)$

$\downarrow$ **GPY sieve** [2] $\downarrow$

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Bounded gaps between primes!

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Work on **large prime gaps**, e.g., **K. Ford, S. Konyagin, J. Maynard, C. Pomerance, and T. Tao** (2018).

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- More precisely, for $(a, d) = 1$, let

$$\Delta(\tau_k; X, d, a) = \sum_{\substack{n \le X \\ n \equiv a \pmod d}} \tau_k(n) - \frac{1}{\varphi(d)} \sum_{\substack{n \le X \\ (n,d)=1}} \tau_k(n).$$

Then, for each k, we seek $\theta_k > 0$ as large as possible such that, for all $\epsilon > 0$, there is $\delta > 0$, such that

$$\Delta(\tau_k; X, d, a) \ll \frac{X^{1-\delta}}{\varphi(d)}$$

for all $d \le X^{\theta_k - \epsilon}$. The number θ_k is called the **level of distribution** for τ_k.

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- **Conjecture (c.f. Elliott-Halberstam)**: $\theta_k = 1$ for all k. GRH $\implies$ $\theta_k < 1/2$ for all k. Moduli $d > X^{1/2}$ are called **large moduli**. This is sometimes refers to as the "**square-root barrier**".

Recall that, for each k, we seek $\theta_k > 0$ as large as possible such that, for each $\epsilon > 0$, there is $\delta > 0$, such that

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- Known results for individual distribution estimate (1) for τ_k. Only for $k = 1, 2, 3$ is the exponent of distribution θ_k for τ_k known to hold for a value larger than $1/2$.

$\boldsymbol{k}$	$\boldsymbol{\theta_k}$	**References**
$k = 1$	$\theta_1 = 1$	See previous slide.
$k = 2$	$\theta_2 = 2/3$	**Selberg, Linnik, Hooley** (independent, unpublished, 1950's).
$k = 3$	$\theta_3 = 1/2 + 1/230$	**Friedlander and Iwaniec** (1985).
	$\theta_3 = 1/2 + 1/82$	**Heath-Brown** (1986).
$k = 4$	$\theta_4 < 1/2$	**Linnik** (1961).
$k \geq 4$	$\theta_k = 8/(3k + 4)$	**Lavrik** (1965).
$k = 5$	$\theta_5 = 9/20$	**Friedlander and Iwaniec** (1985).
$k = 6$	$\theta_6 = 5/12$	**Friedlander and Iwaniec** (1985).
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- This stubborn problem (showing that $\theta_k > 1/2$) has stood static for any $k \geq 4$ ever since the 80's. (Also, the value $\theta_2 = 2/3$ for $\tau(n)$ has not been improved since the 50's.) To make progress, various **averages** on $\Delta(\tau_k; X, d, a)$ are considered by many authors.

- **Y. Zhang** (2014): restrict to **smooth moduli**; in his remarkable work on bounded gaps between primes, a crucial step is to prove, for any $a \neq 0$,

$$\sum_{\substack{d < X^{1/2+2\varpi} \\ (d,a)=1 \\ d \text{ is } X^{\varpi}\text{-smooth}}} \mu(d)^2 |\Delta(\Lambda; X, d, a)| \ll \frac{X}{(\log X)^A}.$$

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- Zhang's method originally for Λ applies equally to τ_k:

Theorem 0 (F. Wei, B. Xue, and Y. Zhang [a] (2016))

[a] F. Wei, B. Xue, and Y. Zhang, Sci. China Math. 59 (2016), **no. 9**, 1663-1668.

For $a \neq 0$, denote

$$\mathcal{D} = \{d \geq 1 : (d,a) = 1, \mu(d)^2, (d, \prod_{p < X^{1/1168}} p) > X^{71/584}\}.$$

Then, for any $k \geq 4$, we have

$$\sum_{\substack{d < X^{293/584} \\ d \in \mathcal{D}}} |\Delta(\tau_k; X, d, a)| \ll_{k,a} X \exp(-\log^{1/2} X).$$

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- We provide a sharpening of this error term.

Our results

Theorem 1 (Main result)

Let $a \neq 0$. *Denote* $\mathcal{P}(y) = \prod_{p<y} p$, $\varpi = 1/1168$, *and*

$$\mathcal{D} = \{d \geq 1 : (d,a) = 1, \mu(d)^2, (d, \mathcal{P}(X^{\varpi^2})) < X^{\varpi}, \text{ and } (d, \mathcal{P}(X^{\varpi})) > X^{1/8-4\varpi}\},$$

Then, for all $k \geq 4$, *we have*

$$\sum_{\substack{d \in \mathcal{D} \\ d < X^{1/2+1/584}}} |\Delta(\tau_k; X, d, a)| \ll X^{1-\theta_k}, \tag{2}$$

where $\theta_k = \min\{1/12(k+2), \varpi^2\}$. *The implied constant is effective and depends on* a *and* k.

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- If we assume the Generalized Lindelöf Hypothesis, we can prove a stronger result.

Theorem 2

On the Generalized Lindelöf Hypothesis, the estimate (2) holds with the right side replaced by

$$X^{1-1/2018^2},$$

where the θ_k *power saving is replaced by a positive constant independent of* k*.*

- If, in addition with averaging over moduli d, we also average over primitive residue classes in each modulus, then we can further extend the range of d.

Theorem 3

For $k \geq 4$ we have

$$\sum_{d \leq D} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \Delta(\tau_k; X, d, a)^2 \ll \begin{cases} X^{2-1/6(k+4)}, & \text{for } 1 \leq D \leq X^{1-1/6(k+2)}, \\ DX(\log X)^{k^2-1}, & \text{for } X^{1-1/6(k+2)} < D \leq X. \end{cases}$$

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- In our last result, we prove a distribution estimate involving Λ where the moduli d can be as large as X^2.

Theorem 4

For $k \geq 4$ there holds

$$\sum_{d \leq D} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} \left(\sum_{\substack{m,n \leq X \\ m \equiv an (\text{mod } d)}} \tau_k(m)\Lambda(n) - \frac{X}{\varphi(d)} \sum_{\substack{n \leq X \\ (n,d)=1}} \tau_k(n) \right)^2$$

$$\ll \begin{cases} X^{4-1/3(k+4)}, & \text{for } 1 \leq D \leq X^{2-1/3(k+2)}, \\ DX^2(\log X)^{2k-2}, & \text{for } X^{2-1/3(k+2)} < D \leq X^2. \end{cases}$$

Thank You!

Figure 1: **Chico State** Motto, "**Today decides tomorrow**".